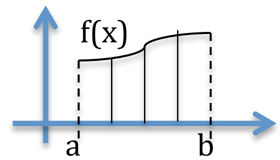
Handout 19

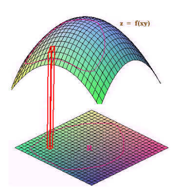
Recall: Given interval [a,b] we divide it into subintervals of equal size Δx, i.e. $x_j = a + j\Delta x, j = 0,1,\ldots,n$ and define:

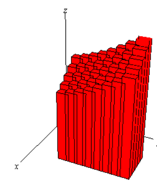
1) Reimann sum: $\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$

2) Integration $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$



Double Integral





For the rectangular domain $[a,b]\times[c,d]$ we divide the rectangle into small sub-rectangles $R_{ij} = [x_{i-1},x_i]\times[y_{i-1},y_i]$ where $x_i = a + i\Delta x, j = 0,1,\ldots,m,\ y_j = c + j\Delta y, j = 0,1,\ldots,n$.
The area of R_{ij} is $\Delta A = \Delta x \cdot \Delta y$.

The double Reimann sums which is also the approximation of the volume under surface z=f(x,y) is given by sums of volumes of the form

$$f\left(x_{ij}^*,y_{ij}^*\right)\Delta A \text{ , i.e. } V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*,y_{ij}^*\right)\Delta A .$$

As usually we take the division of the rectangular into infinitely many infinitely small sub-rectangles to get an exact value of the volume, thus $V = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*,y_{ij}^*\right)\Delta A$

Definition: The double integral of z=f(x,y) over the rectangle R is given by

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*,y_{ij}^*\right)\Delta A \text{ if the limit exists.}$$

Conclusion: $V = \iint_R f(x,y)dA$.

Midpoint Rule: Let $x_i = a + i\Delta x, j = 0,1,\ldots,m,\ y_j = c + j\Delta y, j = 0,1,\ldots,n$ and midpoints

$\bar{x}_i = \frac{x_{i-1}+x_i}{2}, \bar{y}_j = \frac{y_{j-1}+y_j}{2}$ then $\iint_R f(x,y)dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\bar{x}_i,\bar{y}_j\right)\Delta A$.

Average value:

Recall: Given f(x) on an interval [a,b] then $f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$

Given f(x,y) on rectangle R with area A(R) then $f_{ave} = \frac{1}{A(R)}\iint_R f(x,y)\Delta A$

Properties of Double Integrals:

1) $\iint_R f(x,y)+g(x,y)\Delta A=\iint_R f(x,y)\Delta A+\iint_R g(x,y)\Delta A$

2) $\iint_R cf(x,y)\Delta A=c\iint_R f(x,y)\Delta A$

3) $f(x,y)\geq g(x,y)\Rightarrow \iint_R f(x,y)\Delta A\geq \iint_R g(x,y)\Delta A$

Iterated Integrals:

Suppose $f(x,y)$ defined on rectangle $R=[a,b]\times[c,d]$ and let $g(x)=\int_c^d f(x,y)dy$ where the expression $\int_c^d f(x,y)dy$ is understood as a partial integration with respect to y, i.e. the variable x considered a constant for the process of integration. Next we integrate g to get

$$\int_a^b g(x)dx=\int_a^b\left\{\int_c^d f(x,y)dy\right\}dx$$

The last integral is called an **iterated integral;** we often omit the brackets and recognize 2 of them:

1) $\int_a^b\int_c^d f(x,y)dy\,dx=\int_a^b\left\{\int_c^d f(x,y)dy\right\}dx$

2) $\int_c^d\int_a^b f(x,y)dx\,dy=\int_c^d\left\{\int_a^b f(x,y)dx\right\}dy$

Theorem (Fubini's): If *f*(*x*,*y*) is continuous on rectangle $R=[a,b]\times[c,d]$ (or at least bounded with discontinuities on a finite number of smooth curves) then

$$\iint_R f(x,y)dA=\int_a^b\int_c^d f(x,y)dy\,dx=\int_c^d\int_a^b f(x,y)dy\,dx$$

Thm: Let f(x,y) = g(x)h(y) then

$$\int_a^b\int_c^d f(x,y)dy\,dx=\int_a^b\int_c^d g(x)h(y)dy\,dx=\int_a^b g(x)dx\cdot\int_c^d h(y)dy$$

$$\int_c^d\int_a^b f(x,y)dx\,dy=\int_c^d\int_a^b g(x)h(y)dx\,dy=\int_a^b g(x)dx\cdot\int_c^d h(y)dy$$