Handout 18a

Definition: A function of two variables z=f(x,y) has a local maximum at (x,y)=(a,b) if $f(x,y) \le f(a,b)$ when (x,y) is near (a,b). In other words there exists a small disc D centered at (a,b) such that for all $(x,y) \in D$ the inequality $f(x,y) \le f(a,b)$ holds.The value f(a,b) is called local maximum value.
Similarly, f(a,b) is a local minimum if there is a disc centered at (a,b) such that $f(x,y) \ge f(a,b)$ for all $(x,y) \in D$.
Theorem (Fermat's): If z=f(x,y) has a local maximum or minimum at (a,b) and the first order partial derivatives of f exists, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.
Note: This also hints that max/min could be at points where at least one of partial derivatives doesn't exists.

Second Derivative Test: Suppose that z=f(x,y) has continuous partial derivatives on a disc centered at (a,b) and suppose (a,b) is a critical point, that is $f_x(a,b) = 0$ and $f_y(a,b) = 0$. Define $D = D(a,b) = f_{xx}f_{yy} - \left(f_{xy}(a,b)\right)^2$
If $D > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is a local minimum.
If $D > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is a local maximum.
If $D < 0$, then $f(a,b)$ is a saddle point.
If $D = 0$ - the test is indecisive use another method.
Note: It may be easy to remember formula of D it with use of **Hessian Matrix**
$H_{2\times 2} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$, since $D = \det H$.
Definition: An analogous term to a closed $[a,b]$ /open (a,b)/clopen ($[a,b)$ or $(a,b]$) interval in $\mathbb{R}$ is a closed/open/clopen set in $\mathbb{R}^2$ (also in $(\mathbb{R}^n)$). A set is closed if it contains all it's boundary points.
Note: A domain is considered an open set.
Definition: A set S is a bounded set if there is a disc D such that $S \subset D$.

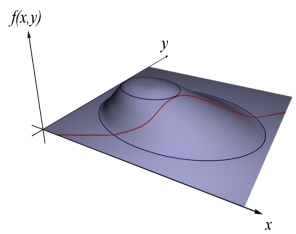
Theorem: **Extreme Value Theorem** for function of two variables z=f(x,y). If f is continuous on a closed bounded set $D \subset \mathbb{R}^2$ then f attains its absolute maximum and minimum values.

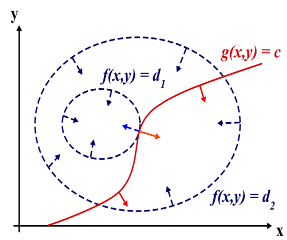
Algorithm: To find the abslolute minimum and maximum values of a continuous function f on a closed bounded set D:
1) Find the values of f at the critical points of f in D.
2) Find the extreme values of f on the boundary of D.
3) The largest/smallest value from 1,2 is the absolute max/min.

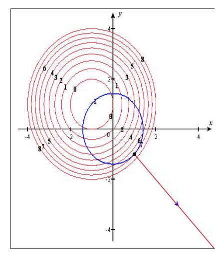
Examples of Max\Min problems with constraint:

1) The tallest student in 1321 class. We have a function that return height of people and the constraint – students of our class.
2) When we looking for max/min of a function in closed bounded set D we have to find min\max on the boundary. Here the constraint here is that we looking for the points along the closed curve which is the boundary of the set D.
3) Consider extreme values on a surface S along a curve C.

Theorem: The extreme value of function z =f(x,y) (z=f(x,y,z)) under constraint g(x,y)=k (g(x,y,z)=k) is at points where *f* and *g* has proportional gradients.







Algorithm: To find the maximum/minimum values of f(x,y)/f(x,y,z) subject to constraint g(x,y)=k/g(x,y,z)=k [assuming that these extreme values exists and $\nabla g \neq 0$ on the surface g=k) .

1) Find (x,y)/(x,y,z) such that $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = k$
   $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z) = k$
2) Evaluate *f* at points you find and order to find min/max

One can also view this problem as a min/max of the Lagrange function

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda \cdot \left(g(x,y,z) - k\right)$$

$$L(x,y,\lambda) = f(x,y) + \lambda \cdot \left(g(x,y) - k\right)$$

The number λ is called Lagrange Multiplier.