Handout 17

Recall: Chain Rule for y=f(x), x=g(t): $\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$

Theorem: Chain Rule for z=f(x,y), x(t)=g(t), y(t)=h(t): $\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$

Theorem: Chain Rule for z=f(x,y), x(t)=g(s,t), y(t)=h(s,t):

$$\frac{\partial z}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \qquad \frac{\partial z}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Theorem: Implicit differentiation

1) Given equation $F(x,y(x))=0$, one differentiate both sides with respect to x to get $F_y\frac{\partial y}{\partial x}+F_x\underbrace{\frac{\partial x}{\partial x}}_{=1}=0$ then solve it for $\frac{dy}{dx}=-\frac{F_x}{F_y}$
2) Similarly, given $F(x,y,z(x,y))=0$:
   a. Differentiating with respect to x gives $F_x\underbrace{\frac{\partial x}{\partial x}}_{=1}+F_y\underbrace{\frac{\partial y}{\partial x}}_{=0}+F_z\frac{\partial z}{\partial x}=0\Rightarrow z_x=-\frac{F_x}{F_z}$
   b. Differentiating with respect to y gives $F_x\underbrace{\frac{\partial x}{\partial y}}_{=0}+F_y\underbrace{\frac{\partial y}{\partial y}}_{=1}+F_z\frac{\partial z}{\partial y}=0\Rightarrow z_y=-\frac{F_y}{F_z}$

Definition: The (Directional) Derivative of a function f (x,y) in a direction of a unit vector $\vec{u}=\langle a,b\rangle$ is given by $D_{\vec{u}}f(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+ah,y_0+bh)a-f(x_0,y_0)}{h}$ if the limit exists.

Theorem: $D_{\vec{u}}f(x_0,y_0)=f_x(x_0,y_0)a+f_y(x_0,y_0)b$

Definition: Gradient vector of a function of 2 variables is defined as

$$\overrightarrow{\nabla f}(x,y)=\langle f_x(x,y),f_y(x,y)\rangle=\frac{\partial f}{\partial x}\text{i}+\frac{\partial f}{\partial y}\text{j}$$

Similarly $\overrightarrow{\nabla f}(x,y,z)=\langle f_x(x,y,z),f_y(x,y,z),f_z(x,y,z)\rangle=\frac{\partial f}{\partial x}\text{i}+\frac{\partial f}{\partial y}\text{j}+\frac{\partial f}{\partial z}\text{k}$

And $\overrightarrow{\nabla f}(x_1,\ldots,x_n)=\langle f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\rangle$

Definition: Another definition of directional derivative:

$$D_{\langle a,b\rangle}f(x,y)=\overrightarrow{\nabla f}(x,y)\cdot\langle a,b\rangle$$
$$D_{\langle a,b,c\rangle}f(x,y,z)=\overrightarrow{\nabla f}(x,y,z)\cdot\langle a,b,c\rangle$$
$$D_{\langle y_1,\ldots,y_n\rangle}f(x_1,\ldots,x_n)=\overrightarrow{\nabla f}(x_1,\ldots,x_n)\cdot\langle y_1,\ldots,y_n\rangle$$

Definition: Yet another way to define directional derivative: let $f:\mathbb{R}^n\to\mathbb{R} f$ be a real valued function with a vector space domain $\mathbb{R}^n$, i.e. $f(\vec{x})\in\mathbb{R}$. Thus, the directional derivative is given by $D_{\vec{u}}f(\vec{x})=\lim_{h\to 0}\frac{f(\vec{x}+h\vec{u})-f(\vec{x})}{h}$

Theorem: The maximum of a directional derivative at given point occurs when the vector u has the same direction as the gradient vector.