Handout 16

Definition: A partial derivative of a function of several variables is a derivative with respect of one of those variables why the other variables consider constant. For a function $f(x,y)$, the partial derivatives are:

$$D_x f = \frac{\partial}{\partial x} f(x,y) = f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$D_y f = \frac{\partial}{\partial y} f(x,y) = f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

For a function $f(x,y,z)$, the partial derivatives are:

$$D_x f = \frac{\partial}{\partial x} f(x,y,z) = f_x(x,y,z) = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

$$D_y f = \frac{\partial}{\partial y} f(x,y,z) = f_y(x,y,z) = \lim_{h \to 0} \frac{f(x,y+h,z) - f(x,y,z)}{h}$$

$$D_z f = \frac{\partial}{\partial z} f(x,y,z) = f_z(x,y,z) = \lim_{h \to 0} \frac{f(x,y,z+h) - f(x,y,z)}{h}$$

Similarly, for a function $f(x_1, x_2, \ldots x_n)$ the partial derivatives $D_1 f = f_1, D_2 f = f_2, \ldots, D_n f = f_n$

Higher derivatives

Partial derivatives of a function of several variables are functions of several variables, and therefore, may have their partial derivatives. We denote it as:

$$f_{xx} = \frac{\partial f}{\partial x^2} \qquad f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f}{\partial y \partial x} \qquad f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x \partial y} \qquad f_{yy} = \frac{\partial f}{\partial y^2}$$

Theorem: If f_{xy}, f_{yx} continuous on a disc around (a,b) then $f_{xy}, (a,b) = f_{yx}(a,b)$

Definition: Partial differential equations are differential equations that involve partial derivatives.

- Helmholtz equation $u_{xx} + u_{yy} + k^2 u = 0$.
- Laplace equation $u_{xx} + u_{yy} = 0$
- Heat equation $u_t = a^2 u_{xx}$
- Wave equation $u_{tt} = a^2 u_{xx}$

Note: A plane $\vec{n}(r-r_0)=\tilde{A}(x-x_0)+\tilde{B}(y-y_0)+\tilde{C}(z-z_0)=0$ can be described as $z-z_0=A(x-x_0)+B(y-y_0)$, where $A=\tilde{A}/\tilde{C},\ B=\tilde{B}/\tilde{C}$

Definition: Let S be surface described by a function z=f(x,y) and suppose that *f* has continuous partial derivatives. An equation of the tangent plane to S at point $P(x_0,y_0,z_0)$ is given by: $z-z_0=f_x(x-x_0)+f_y(y-y_0)$
Similarly a surface defined by f(x,y,z)=0 the tangent plane is
$f_x(x-x_0)+f_y(y-y_0)+f_z(z-z_0)=0$

Definition: A tangent plane to parametric surface $\vec{r}(u,v)=x(u,v)\mathrm{i}+y(u,v)\mathrm{j}+z(u,v)\mathrm{k}$ is given using a normal vector $\vec{n}=\vec{r}_u\times\vec{r}_v=\langle x_u,y_u,z_u\rangle\times\langle x_v,y_v,z_v\rangle$.

Definition: Suppose that a function *f*(*x*,*y*) has continuous partial derivatives. A function $L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$ is called **linearization of *f***.
A **linear approximation** or a **tangent plane approximation** of *f* at (a,b) is defined by

$$f(x,y)\approx f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)=L(x,y) \text{ or equivalently}$$

$$f(a+h_x,b+h_y)\approx f(a,b)+f_x(a,b)h_x+f_y(a,b)h_y$$

We defined linear approximation for function with continuous partial derivatives, however what happens when the derivatives aren't continuous?

Definition: The function z=f(x,y) is **differentiable** at (a,b) if $\Delta z=\Delta f=f(a+\Delta x,b+\Delta y)-f(a,b)$ can be expressed in the form

$$\Delta z=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\varepsilon_1(\Delta x,\Delta y)\Delta x+\varepsilon_2(\Delta x,\Delta y)\Delta y$$

where $\lim_{(\Delta x,\Delta y)\to 0}\varepsilon_1(\Delta x,\Delta y)=0$ and $\lim_{(\Delta x,\Delta y)\to 0}\varepsilon_2(\Delta x,\Delta y)=0$
or equivalently $\Delta z=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\varepsilon(\Delta x,\Delta y)\sqrt{\Delta x^2+\Delta y^2}$ where $\lim_{(\Delta x,\Delta y)\to 0}\varepsilon(\Delta x,\Delta y)=0$. The second form is more convenient for calculations, since one solves it for

$$\varepsilon(\Delta x,\Delta y)=\frac{f(a+\Delta x,b+\Delta y)-f(a,b)-f_x(a,b)\Delta x-f_y(a,b)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}$$

Theorem: If the partial derivatives of *f* exists near (a,b) and continuous at (a,b) then f is differentiable.

Definition: An analogy to the differentials in 1D $dy=f'(x)dx$ is called a **total differential** in 2D and 3D and is given by $df=f_x(x,y)dx+f_y(x,y)dy$ and $df=f_x(x,y,z)dx+f_y(x,y,z)dy+f_z(x,y,z)dz$ respectively.