Handout 14

Definition: An **Arc Length** of a curve described by vector function $\vec{v}(t)=\langle v_1(t),v_2(t),v_3(t)\rangle$ for $a\le t\le b$ is defined by

$$L=\int_a^b\left|\vec{v}\,'(t)\right|dt=\int_a^b\sqrt{v'_1(t)^2+v'_2(t)^2+v'_3(t)^2}\,dt\,.$$

Note: this is similar to the definition we learned in calculus I for parameterized curve in 2D.

Arc length parameterization

The same curve can be described by different parameterizations, for example $\langle t,\sqrt{t},\sin t\rangle$ for $0\le t\le\pi$ describe the same curve as $\langle \pi t,\sqrt{t\pi},\sin\pi t\rangle$ for $0\le t\le 1$. A natural parameterization for a space curve is with respect to arc length. Thus, if a curve is parameterized with parameter t, i.e. as $\vec{r}(t)$, then we can find an arc length function as $s(t)=\int_0^t\left|\vec{r}\,'(t)\right|dt$ and try to re-parameterize it using $t=t(s)$. The distance along a space curve is independent of parameterization. This simply means that the total distance traveled along a curve is independent of the speed.

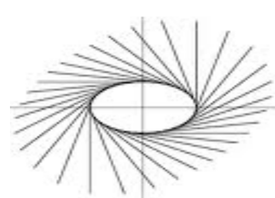
Note: It is usually difficult to find explicit formula for an arc length parameterization of a general curve. Fortunately, it can be used and have many applications even without explicit formula.

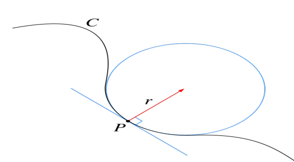
Definition: A parameterization $\vec{r}(t)$ is called smooth on an interval I if $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t)\neq 0$ on I. A curve is called smooth if it has a smooth parameterization.

Definition: The **unit tangent vector** of a smooth curve $\vec{r}(t)$ is defined by

$$\vec{T}(t)=\frac{\vec{r}\,'(t)}{\left|\vec{r}\,'(t)\right|}$$

Note: $\vec{T}(t)$ changes direction slowly when the curve is straight\flat and when the curve is sharp it change the direction faster.





Definition: A curvature of a curve $\vec{r}(t)$ is measure how quickly the curve change direction at given point. A curvature can be understood as a reciprocal radius of inscribed\osculated circle tangent to the curve, also called circle of curvature. The curvature is often denoted as κ (kappa) and it given by $\kappa = \left|\frac{d\vec{T}}{ds}\right|$ where $\vec{T}(t)$ is a **unit tangent vector** and s is arc length parameter (so the curvature is independent of parameterization).

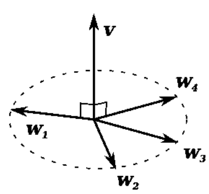
Since $\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt}$, one writes $\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}/dt}{ds/dt}\right|$. The derivative of the formula of the arc length parameter $s(t) = \int_0^t |\vec{r}\,'(t)|dt$ gives $\frac{ds}{dt} = |\vec{r}\,'(t)|$, therefore we reformulate the curvature as $\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$

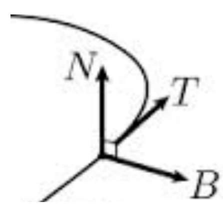
Theorem: $\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

The Normal and Binormal Vectors

There is more then one vector orthogonal to the tangent vector $\vec{T}(t)$. We already found that $\vec{T}\,'(t)$ is orthogonal to $\vec{T}(t)$. Note that $\vec{T}\,'(t)$ isn't unit vector. We define a (**principle**) **unit normal vector** to be $\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$. We define additional vector, binormal vector by $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ which is orthogonal to both $\vec{T}(t)$ and $\vec{N}(t)$, and is also a unit vector, since $|\vec{B}(t)| = |\vec{T}(t)||\vec{N}(t)| = 1$.

A plane determined by $\vec{N}(t)$ and $\vec{B}(t)$ is called normal plane, a plane determined by $\vec{T}(t)$ and $\vec{N}(t)$ is called osculating plane and is related to the circle of curvature.





Definition: Let t represent a time and $\vec{r}(t)$ a trajectory of a moving particle. Then the velocity vector is defined by $\vec{v}(t) = \vec{r}\,'(t)$, the speed is given by $v = |\vec{v}(t)| = |\vec{r}\,'(t)|$ and the acceleration reads by $\vec{a} = |\vec{v}\,'(t)| = |\vec{r}\,''(t)|$.

Theorem: The acceleration of a particle following the curve $\vec{r}(t)$ consist of 2 components a change of speed $a_T = \frac{d}{dt}|\vec{v}(t)|$ and a change of velocity direction $a_N = \kappa|\vec{v}(t)|^2$, so that $\vec{a} = a_T\vec{T}(t) + a_N\vec{N}(t)$.