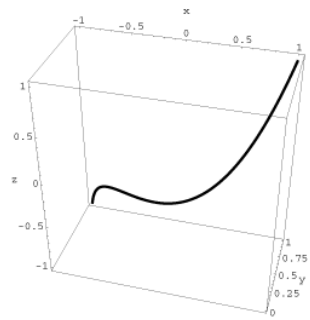
# Handout 13

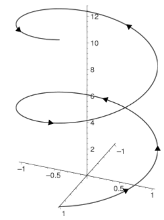
**Definition**: A vector-valued function is a function whose domain is a set of real numbers and a range is a set of vectors. We denote it as $f: R \to R^2$ if a range of the function is set of 2D vectors or $f: R \to R^3$ if the range is set of 3D vectors.

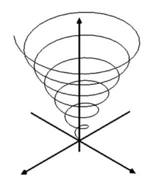
**Note**: we are most interested in $f: R \to R^3$

**Examples**:

1. General case: $r(t) = \langle f(t), g(t), h(t) \rangle$
2. A linear function or a line: $\vec{l}(t) = \langle a_1 t + b_1, a_2 t + b_2, a_3 t + b_3 \rangle$
3. Parabola in xy plane $r(t) = \langle t, t^2, 0 \rangle$
4. A twisted cubic: $r(t) = t\,\mathrm{i} + t^2\,\mathrm{j} + t^3\,\mathrm{k}$
5. A helix $r(t) = \cos t\,\mathrm{i} + \sin t\,\mathrm{j} + t\,\mathrm{k}$
6. An opposite direction, different start point helix $r(t) = \sin t\,\mathrm{i} + \cos t\,\mathrm{j} + t\,\mathrm{k}$
7. Spiral along y-axis $r(t) = \sin \mathrm{i} + t\,\mathrm{j} + \cos t\,\mathrm{k}$
8. Tornado $\langle t\cos t, t\sin t, t \rangle$





**Definition**: A limit of vector-valued function $r(t) = \langle f(t), g(t), h(t) \rangle$ is given by $\lim_{t \to a} r(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$ provided the limits of the component function exist.

**Definition**: Derivative of a vector function is defined by $\frac{d\vec{f}}{dt} = \vec{f}\,'(t) = \lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}$

**Theorem**: $\vec{f}\,'(t) = \frac{d}{dt} \langle f_1(t), f_2(t), f_3(t) \rangle = \langle f_1'(t), f_2'(t), f_3'(t) \rangle$

**Definition**: A vector tangent to a curve $\vec{r}(t)$ at point $t_0$ is given by $\vec{r}\,'(t_0)$ provided that $\vec{r}(t)$ exists at $t_0$ and that $\vec{r}\,'(t_0) \neq 0$. We often interested in unit tangent vector defined by $T(t_0) = \frac{\vec{r}\,'(t_0)}{|\vec{r}\,'(t_0)|}$.

**Differentiating rules:**

1. Derivatives Sum of vector functions $\left(\vec{u}(t)+\vec{v}(t)\right)'=\vec{u}'(t)+\vec{v}'(t)$
2. Derivative of a scalar times vector function $\left(c\vec{u}(t)\right)'=c\vec{u}'(t)$
3. Derivative of a scalar function times vector function

$$\begin{aligned}\left(f(t)\vec{v}(t)\right)' &= \frac{d}{dt}\langle f(t)v_1(t),f(t)v_2(t),f(t)v_3(t)\rangle = \\ &= \langle f'(t)v_1(t)+f(t)v'_1(t),f'(t)v_2(t)+f(t)v'_2(t),f'(t)v_3(t)+f(t)v'_3(t)\rangle \\ &= f'(t)\vec{v}(t)+f(t)\vec{v}'(t)\end{aligned}$$

4. Dot product rule

$$\begin{aligned}\left(\vec{u}(t)\cdot\vec{v}(t)\right)' &= \frac{d}{dt}\left(u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t)\right) = \\ &= u'_1(t)v_1(t)+u_1(t)v'_1(t)+u'_2(t)v_2(t)+u_2(t)v'_2(t)+u'_3(t)v_3(t)+u_3(t)v'_3(t) = \\ &= \vec{u}'(t)\cdot\vec{v}(t)+\vec{u}(t)\cdot\vec{v}'(t)\end{aligned}$$

5. Cross product rule

$$\begin{aligned}\left(\vec{u}(t)\times\vec{v}(t)\right)' &= \frac{d}{dt}\langle u_2(t)v_3(t)-u_3(t)v_2(t),\bullet,\bullet\rangle = \\ &= \langle u'_2(t)v_3(t)+u_2(t)v'_3(t)-\left(u'_3(t)v_2(t)+u_3(t)v'_2(t)\right),\bullet,\bullet\rangle \\ &= \vec{u}'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}'(t)\end{aligned}$$

6. Chain rule:

$$\begin{aligned}\left(\vec{u}(f(t))\right)' &= \frac{d}{dt}\langle u_1(f(t)),u_2(f(t)),u_3(f(t))\rangle = \\ &= \langle u'_1(f(t))f'(t),u'_2(f(t))f'(t),u'_3(f(t))f'(t)\rangle = \\ &= \vec{u}'(f(t))f'(t)\end{aligned}$$

## Integration of vector function

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b r_1(t)\,dt, \int_a^b r_2(t)\,dt, \int_a^b r_3(t)\,dt \right\rangle$$