Handout 10

Definition: A 2x2 matrix is a mathematical entity of the following form $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Similarly a 3x3 matrix has a form $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Definition: Every matrix is associated with a special number called **determinant**. The determinant of 2x2 matrix is given by $\det A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$. The determinant of 3x3 matrix is given in terms of determinants of 2x2 matrices (called minors, and denoted M_{ij}) as following

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| =$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} =$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Hint: The Minor M_{ij} obtained by erasing the ith column and jth row of the matrix A

$$M_{11} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}; \quad M_{12} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}; \quad M_{13} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Definition: The cross product between vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is given by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} =$$

$$= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Notes:

- cross product is not commutative, i.e. $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$
- $\vec{a} \times \vec{a} = \langle a,b,c \rangle \times \langle a,b,c \rangle = \langle a_2a_3 - a_3a_2, a_3a_1 - a_1a_3, a_1a_2 - a_2a_1 \rangle = \vec{0}$
- if $b = k\vec{a}$ then $\vec{a} \times \vec{b} = 0$
- $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

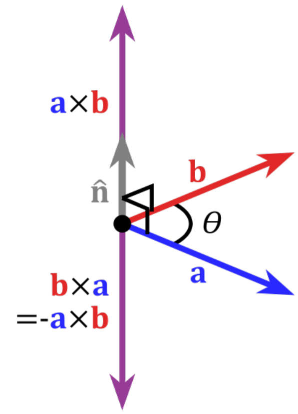
Properties of Cross Product

1) $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$

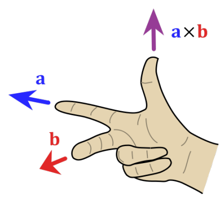
2) $(k\mathbf{a})\times\mathbf{b}=k(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times(k\mathbf{b})$

3) $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$

4) $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$

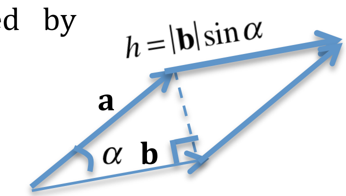


Definition: Geometrical definition of cross product is given by $\vec{a}\times\vec{b}=\left(|\vec{a}||\vec{b}|\sin\alpha\right)\mathbf{n}$ where α is the angle between the vectors $\vec{a}$ and $\vec{b}$ and $\boldsymbol{n}$ is a unit vector perpendicular\orthogonal to both vectors $\vec{a}$ and $\vec{b}$.



Corollary: $\vec{a}\times\vec{b}$ orthogonal to both vectors $\vec{a}$ and $\vec{b}$.
Corollary: Non zero vectors $\vec{a}$ and $\vec{b}$ are parallel iff $\vec{a}\times\vec{b}=0$.

Theorem: $\left|\vec{a}\times\vec{b}\right|=\left||\vec{a}||\vec{b}|\sin\alpha\right|$ is area of parallelogram determined by vectors $\vec{a}$ and $\vec{b}$.



Definition: An angle between 2 vectors defined $|\sin\alpha|=\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$. Thus

$$0\le\alpha\le\frac{\pi}{2}:\alpha=\arcsin\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\qquad -\frac{\pi}{2}\le\alpha\le 0:\alpha=-\arcsin\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

Definition: **Scalar Triple product** of vectors $\vec{a},\vec{b},\vec{c}$ is defined to be $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$.

If we think about parallelepiped defined by vectors $\vec{a},\vec{b},\vec{c}$ with with a base of parallelogram defined by $\vec{b},\vec{c}$. The height of the parallelogram is $h=|\vec{a}\cos\theta|$ and the area of the base is $A=\left|\vec{b}\times\vec{c}\right|$.

Definition: The volume parallelepiped defined by vectors $\vec{a},\vec{b},\vec{c}$ is given by:

$$V=hA=|\vec{a}|\cos\theta\left|\vec{b}\times\vec{c}\right|=\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\,.$$

Definition: The **vector triple product is** given by $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=(\vec{a}\cdot\vec{c})\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$