Handouts 6

Theorem: If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence R>0, then the function $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ is continues, differentiable and integrable on the interval $(a-R, a+R)$ and

$$f'(x)=\frac{d}{dx}\sum_{n=1}^{\infty} c_n (x-a)^n=\sum_{n=1}^{\infty}\frac{d}{dx}c_n (x-a)^n=\sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$$

$$\int f(x)dx=\int\sum_{n=0}^{\infty} c_n (x-a)^n dx=\sum_{n=0}^{\infty}\int c_n (x-a)^n dx=\sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}+C$$

with the same radius of convergence R. However the convergence at end points may change.

Note: The theorem above state that the power series can be integrated/differentiated term-by-term. However, for other types of series of functions ($\sum_{n=0}^{\infty} c_n g_n(x)$) the situation is not as simple.

Taylor and Maclaurin Series (8.7)

Example: Consider that a function f has a representation as power series in some interval $0<|x-a|<R$. $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\ldots$.

Note:
$$\begin{aligned} f(a) &= c_0+c_1(a-a)+c_2(a-a)^2+c_3(a-a)^3+\ldots &&= c_0 \\ f'(a) &= c_1+2c_2(a-a)+3c_3(a-a)^2+\ldots &&= c_1 \\ f''(x) &= 2c_2+2\cdot 3c_3(a-a)+3\cdot 4c_4(a-a)^2+\ldots &&= 2c_2 \\ f^{(n)}(a) &= 2\cdot 3\cdot 4\cdots nc_n &&= n! \end{aligned}$$
, thus $c_n \Rightarrow c_n=\frac{f^{(n)}(a)}{n!}$

Theorem: If a function f has a power series representation (expansion) about a, that is $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$, then $c_n=\frac{f^{(n)}(a)}{n!}$, i.e. $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$.

Definition: the series $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ are called **Taylor series/expansion** of the function f (about/centered at) at a. For a special case a=0 the series $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ called **Maclaurin series.**

Definition: Taylor\Maclaurin polynomial is a partial sum of Taylor\Maclaurin series, i.e. $T_m = \sum_{n=0}^{m} \frac{f^{(n)}(0)}{n!} x^n$. The remainder is defined as $R_m(x) = f(x) - T_m(x)$

Theorem: If $f(x) = T_m(x) + R_m(x)$ and $\lim_{m\to\infty} R_m(x) = 0, \forall x \in (a-R, a+R)$ then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \forall x \in (a-R, a+R).$$

Furthermore, if $\max_{|x-a|<d} \left|f^{(m+1)}(x)\right| = M$ (i.e. $\left|f^{(m+1)}(x)\right| \le M, \forall |x-a| < d$), where $d > 0$ is small positive number then $\left|R_m(x)\right| \le \frac{M}{(m+1)!}|x-a|^{m+1}, \forall |x-a| < d$

Definition: The numbers $\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$ called binomial coefficients.

Important Maclaurin Series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, -\infty < x < \infty$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, -\infty < x < \infty$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, -\infty < x < \infty$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots, -1 < x < 1$$

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, -1 < x \le 1$$

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, -1 \le x \le 1$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots + \frac{m(m-1)\cdot\ldots\cdot(m-k+1)}{k!}x^k + \ldots, m \in R,$$

$m \ge 0, \Rightarrow -1 \le x \le 1$, $-1 < m < 0, \Rightarrow -1 < x \le 1$, $m \le -1, \Rightarrow -1 < x < 1$.