Handout 4

Definition: A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

Theorem: If a series $\sum a_n$ is absolutely convergent then it is convergent.

It is true because 1) $0 \le a_n + |a_n| \le 2|a_n|$, 2) $\sum |a_n|$ is convergent and so $2\sum |a_n|$ and by comparison test ($\sum (a_n + |a_n|) \le 2\sum |a_n|$) also $\sum a_n + |a_n|$ is convergent. Finally

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

Definition: A series $\sum a_n$ is called conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

The ratio test theorem:

- If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then the series $\sum a_n$ is absolutely convergent (and therefore convergent)
- If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ (including $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$) then the series $\sum a_n$ divergent.
- If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ then the ratio test inconclusive, that is we have to use another method to determine convergence or divergence of $\sum a_n$.

Theorem: Let $a_k = f(k)$, where $f(x)$ is a continues, positive, decreasing function for $x \ge n$ (these are conditions for a integral test). Consider that $S = \sum_{n=1}^{\infty} a_n$ is convergent series. Let $S_m = \sum_{n=1}^{m} a_n$ be a partial sum and $R_m = S - S_m = \sum_{n=m+1}^{\infty} a_n$ is the remainder, then

$$\int_{m+1}^{\infty} f(x) \le R_m \le \int_{m}^{\infty} f(x).$$

Theorem: If an alternating series $S = \sum b_n = \sum (-1)^n a_n$ satisfy $a_{n+1} \le a_n$ and $\lim\limits_{n\to\infty} a_n = 0$ then $|R_m| = |S - S_m| \le b_n$