Handout 3

The Comparison Test: Let $\sum a_n, \sum b_n$ be series with $a_n, b_n > 0$, then

- If $\sum b_n$ is convergent and $a_n \le b_n$ then $\sum a_n$ is also convergent
- If $\sum b_n$ is divergent and $a_n \ge b_n$ then $\sum a_n$ is also divergent

The Limit Comparison Test: Let $\sum a_n, \sum b_n$ be series with $a_n, b_n > 0$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where $c > 0$ is a finite constant, then either both series converge or both diverge.

Definition: Let $a_n > 0$, then

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - + \ldots$$
$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + - \ldots$$

called alternating series.

Theorem: If an alternating series, either

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - + \ldots \text{ or}$$

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + - \ldots,$$

where $a_n > 0$ satisfy 1) $b_{n+1} \le b_n$ and 2) $\lim_{n\to\infty} b_n = 0$. Then the series are converges.

Definition: A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

Theorem: If a series $\sum a_n$ is absolutely convergent then it is convergent.

It is true because 1) $0 \le a_n + |a_n| \le 2|a_n|$, 2) $\sum |a_n|$ is convergent and so $2\sum |a_n|$ and by comparison test ($\sum (a_n + |a_n|) \le 2\sum |a_n|$) also $\sum a_n + |a_n|$ is convergent. Finally

$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$