# Handout 1

**Definition**: A **sequence** (or an **infinite sequence**) is a function $f:\mathbb{N}\to\mathbb{R}$ that often given as $f(n)=a_n$. We will often write sequences as $\{a_n\}_{n=1}^{\infty}=\{a_n\}_{n\in\mathbb{N}}$.

**Definition**: A **subsequence** is a sequence that can be derived from another sequence by deleting some elements without changing the order of the remaining elements.

**Definition**: A sequence $\{a_n\}$ has a limit L, i.e. $\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L$ as $n\to\infty$ if we can make the terms $a_n$ as close to L as we like by taking n sufficiently large. If $\lim\limits_{n\to\infty} a_n$ exists we say the sequence converges (convergent). Otherwise the sequence is diverges (divergent).

**Thm**: If $\lim\limits_{x\to\infty} f(x)=L$, then the sequence $f(n)=a_n$ is convergent and $\lim\limits_{n\to\infty} a_n = L$

**Limits properties**:

If $a_n, b_n$ are convergent and c is constant then

1) $\lim\limits_{n\to\infty}(a_n \pm b_n)=\lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$
2) $\lim\limits_{n\to\infty} ca_n = c\lim\limits_{n\to\infty} a_n$ including $\lim\limits_{n\to\infty} c = c$
3) $\lim\limits_{n\to\infty}(a_n b_n)=\lim\limits_{n\to\infty} a_n \lim\limits_{n\to\infty} b_n$
4) $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}, \lim\limits_{n\to\infty} b_n \neq 0$
5) $\lim\limits_{n\to\infty} a_n^p = \left(\lim\limits_{n\to\infty} a_n\right)^p, p>0, a_n>0$

**The squeeze theorem for sequences**: If $a_n \le b_n \le c_n$ and $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$ then $\lim\limits_{n\to\infty} b_n = L$

**Absolute Value Theorem**: $\lim\limits_{n\to\infty}|a_n| = 0 \Rightarrow \lim\limits_{n\to\infty} a_n = 0$ (Since $-|a_n| \le a_n \le |a_n|$)

**Thm**: If *f* is continuous function at L and $\lim\limits_{n\to\infty} a_n = L$ then $\lim\limits_{n\to\infty} f(a_n) = f\left(\lim\limits_{n\to\infty} a_n\right) = f(L)$

**Thm**: If a sequence $\{a_n\}_{n=1}^{\infty}$ converges iff subsequences $\{a_{2n}\}_{n=1}^{\infty}$ and $\{a_{2n+1}\}_{n=1}^{\infty}$ does.

**Thm**: If a sequence $\{a_n\}_{n\in\mathbb{N}}^{\infty}$ converges iff all its subsequences converges.

**Corollary**: If there exists a divergent subsequence of $\{a_n\}_{n\in\mathbb{N}}^{\infty}$, then $\{a_n\}_{n\in\mathbb{N}}^{\infty}$ diverges.

**Theorem**: $\lim_{n\to\infty} r^n = \begin{cases} 0 & -1 < r < 1 \\ 1 & r = 1 \end{cases}$, When $r > 1$ the sequence tends to infinity, and it doesn't exists when $r < -1$ (the last 2 are divergent sequences).

**Definition**: A sequence $\{a_n\}$ is increasing if $a_n \le a_{n+1}$ for all $n \ge 1$. It is called decreasing if it is $a_n \ge a_{n+1}$ for all $n \ge 1$. A sequence is monotonic if it is either increasing or decreasing.

**Definition**: A sequence $\{a_n\}$ is bounded above if there is number M such that $a_n \le M, \forall n \ge 1$. It is bounded below if there is number m such that $a_n \ge m, \forall n \ge 1$. If it is bounded above and below it called bounded sequence.

**Theorem**: Every bounded, monotonic sequence is convergent.

**Theorem:** If $\{b_n\}$ is a subsequence of sequence $\{a_n\}$ obtained by deletion of its first $n_0$ (finite number) terms. Then $\{a_n\}$ converges iff $\{b_n\}$ does.

**Monotonicity tests**: 1) $\operatorname{sgn}(a_{n+1} - a_n)$ 2) Does $\frac{a_{n+1}}{a_n} < 1$ or $\frac{a_{n+1}}{a_n} > 1$?