Average value:

Recall: Given f(x) on an interval [a,b] then $f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$

Given f(x,y) on rectangle R with area A(R) then $f_{ave} = \frac{1}{A(R)}\iint_R f(x,y)\Delta A$

Ex 2. Let $z = x^2 - y$ be a shape of a "sand dune" in a box with a base of a 2x2 units size. Answer the following question using the course approximation of the volume of the sand from previous exercise. Could the entire dune be packed if a box has a height of 1 unit?

In previous question we found that the volume of $z = x^2 - y$ on 2x2 square is approximately one, while height of the box should of average value, which is given by $f_{ave} = \frac{V}{A(R)} \approx \frac{1}{4}$. Therefore we will have more then enough place for our sand in that box.

Properties of Double Integrals:

1) $\iint_R f(x,y) + g(x,y)\Delta A = \iint_R f(x,y)\Delta A + \iint_R g(x,y)\Delta A$

2) $\iint_R cf(x,y)\Delta A = c\iint_R f(x,y)\Delta A$

3) $f(x,y) \ge g(x,y) \Rightarrow \iint_R f(x,y)\Delta A \ge \iint_R g(x,y)\Delta A$

5.2 Iterated Integrals (12.2)

It won't be easy to evaluate double integral using the definition with the limits. In Calculus I The Fundamental Theorem of Calculus provided a more convenient way. In this section we will learn an easy method to solve double integrals.

Suppose $f(x,y)$ defined on rectangle $R = [a,b]\times[c,d]$ and let $g(x) = \int_c^d f(x,y)dy$ where the expression $\int_c^d f(x,y)dy$ is understood as a partial integration with respect to y, i.e. the variable x considered a constant for the process of integration. Next we integrate g to get

$$\int_a^b g(x)dx = \int_a^b \left\{ \int_c^d f(x,y)dy \right\} dx$$

The last integral is called an **iterated integral;** we often omit the brackets and recognize 2 of them:

1) $\int_a^b \int_c^d f(x,y)dy\,dx = \int_a^b \left\{ \int_c^d f(x,y)dy \right\} dx$ 2) $\int_c^d \int_a^b f(x,y)dx\,dy = \int_c^d \left\{ \int_a^b f(x,y)dx \right\} dy$

Ex 3. $$\iint_R ye^x dA = \int_{x=-1}^{1}\int_{y=0}^{1} ye^x\,dy\,dx = \int_{-1}^{1} \frac{y^2}{2}e^x\Big|_0^1 dx = \frac{1}{2}\int_0^1 e^x\,dx = \frac{1}{2}e^x\Big|_{-1}^{1} = \frac{e^1 - e^{-1}}{2}$$

Ex 4. $$\int_{x=0}^{1}\int_{y=0}^{x} y^2 x\,dy\,dx = \int_0^1 \left(\frac{y^3}{3}\cdot x\right)_o^x dx = \int_0^1 \frac{x^4}{3}dx = \frac{x^5}{15}\Big|_0^1 = \frac{1}{15}$$

Ex 5. $$\int_{x=0}^{1}\int_{y=0}^{2} x + 3y^2 + 2xy\,dy\,dx = \int_0^1 \left(xy + y^3 + \frac{1}{2}xy^2\right)_0^2 dx =$$
$$= \int_0^1 2x + 2^3 + \frac{1}{2}x2^2\,dx = 2\int_0^1 2x + 4\,dx = 2\left(x^2 + 4x\right)_0^1 = 10$$

Ex 6. $$\int_{y=0}^{2}\int_{x=0}^{1} x + 3y^2 + 2xy\,dx\,dy = \int_{y=0}^{2}\left(\frac{1}{2}x^2 + 3y^2x + x^2y\right)_0^1 dy =$$
$$= \int_{y=0}^{2} \frac{1}{2} + 3y^2 + y\,dy = \left(\frac{1}{2}y + y^3 + \frac{1}{2}y\right)_0^2 = \left(y + y^3\right)_0^2 = 2 + 8 = 10$$

Thm (Fubini's): If $f(x,y)$ is continuous on rectangle $R = [a,b]\times[c,d]$ (or at least bounded with discontinuities on a finite number of smooth curves) then
$$\iint_R f(x,y)dA = \int_a^b\int_c^d f(x,y)dy\,dx = \int_c^d\int_a^b f(x,y)dy\,dx$$

We won't prove this theorem, but to make some sense, recall that $g(x_0) = \int_c^d f(x_0,y)dy$ represents area under curve described by $f(x_0,y)$ and therefore the volume is given by $\iint_R f(x,y)dA = V = \int_a^b g(x)dx = \int_a^b\int_c^d f(x,y)dy\,dx$. Similarly for $\int_c^d\int_a^b f(x,y)dx\,dy$, just x and y change roles.

Thm: Let f(x,y) = g(x)h(y) then
$$\int_a^b\int_c^d f(x,y)dy\,dx = \int_a^b\int_c^d g(x)h(y)dy\,dx = \int_a^b g(x)dx\cdot\int_c^d h(y)dy$$
$$\int_c^d\int_a^b f(x,y)dx\,dy = \int_c^d\int_a^b g(x)h(y)dx\,dy = \int_a^b g(x)dx\cdot\int_c^d h(y)dy$$

Ex 7. $$\int_{x=0}^{1}\int_{y=0}^{1} yx\,dy\,dx = \int_{x=0}^{1}\left(\frac{y^2}{2}x\right)_0^1 dx = \int_{x=0}^{1}\frac{1}{2}x\,dx = \left(\frac{1}{2}\frac{x^2}{2}\right)_0^1 = \frac{1}{4}$$
$$\int_{x=0}^{1}\int_{y=0}^{1} yx\,dy\,dx = \int_{x=0}^{1} x\,dx\cdot\int_{y=0}^{1} y\,dy = \left(\frac{y^2}{2}\right)_0^1\cdot\left(\frac{y^2}{2}\right)_0^1 = \frac{1}{4}$$