

4.7 Maximum and Minimum Values (11.7)

Def: A function of two variables $z=f(x,y)$ has a local maximum at $(x,y)=(a,b)$ if $f(x,y) \leq f(a,b)$ when (x,y) is near (a,b) . In other words there exists a small disc D centered at (a,b) such that for all $(x,y) \in D$ the inequality $f(x,y) \leq f(a,b)$ holds. The value $f(a,b)$ is called local maximum value. Similarly, $f(a,b)$ is a local minimum if there is a disc centered at (a,b) such that $f(x,y) \geq f(a,b)$ for all $(x,y) \in D$.

Thm (Fermat's): If $z=f(x,y)$ has a local maximum or minimum at (a,b) and the first order partial derivatives of f exists, then $f_x(a,b)=0$ and $f_y(a,b)=0$.

Note: This also hints that max/min could be at points where at least one of partial derivatives doesn't exists.

Second Derivative Test: Suppose that $z=f(x,y)$ has continuous partial derivatives on a disc centered at (a,b) and suppose (a,b) is a critical point, that is $f_x(a,b)=0$ and $f_y(a,b)=0$. Define

$$D = D(a,b) = f_{xx}f_{yy} - (f_{xy}(a,b))^2$$

If $D > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is a local minimum.

If $D > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is a local maximum.

If $D < 0$, then $f(a,b)$ is a saddle point.

If $D = 0$ - the test is indecisive use another method.

Note: It may be easy to remember formula of D it with use of **Hessian Matrix**

$$H_{2 \times 2} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}, \text{ since } D = \det H.$$

Def: An analogous term to a closed $[a,b]$ /open (a,b) /clopen $([a,b)$ or $(a,b]$) interval in \mathbb{R} is a closed/open/clopen set in \mathbb{R}^2 (also in (\mathbb{R}^n)). A set is closed if it contain all it's boundary points.

Note: A domain is considered an open set.

Def: A set S is a bounded set if there is a disc D such that $S \subset D$.

Thm: Extreme Value Theorem for function of two variables $z=f(x,y)$. If f is continuous on a closed bounded set $D \subset \mathbb{R}^2$ then f attains its absolute maximum and minimum values.

Alg: To find the absolute minimum and maximum values of a continuous function f on a closed bounded set D :

- 1) Find the values of f at the critical points of f in D .
- 2) Find the extreme values of f on the boundary of D .
- 3) The largest/smallest value from 1,2 is the absolute max/min.

Ex 8. Find and classify all critical points of $f(x,y) = (x-y)e^{xy}$ in \mathbb{R}^2

The function is differential everywhere (as a product of differential functions), so the critical points are only at zeros of partial derivatives.

$$\begin{cases} f_x = e^{xy} + (x-y)ye^{xy} = e^{xy}(1+xy-y^2) = 0 \\ f_y = -e^{xy} + (x-y)xe^{xy} = e^{xy}(-1+x^2-xy) = 0 \end{cases} \Rightarrow \begin{cases} 1+xy-y^2 = 0 \\ 1+xy-x^2 = 0 \end{cases} \Rightarrow \begin{cases} x = (y^2-1)/y \\ 1+(y^2-1) - \frac{(y^2-1)^2}{y^2} = 0 \end{cases}$$

thus $x = -y = \pm 1/\sqrt{2}$ and a critical points are $(1/\sqrt{2}, -1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2})$.

Calculate $D = f_{xx}f_{yy} - f_{xy}^2$:

$$f_{xx} = ye^{xy}(2+xy-y^2) \quad f_{yy} = xe^{xy}(-2+x^2-xy) \quad f_{xy} = e^{xy}(2x+x^2y-xy^2-2y)$$

$$D = yxe^{2xy}(2+xy-y^2)(-2+x^2-xy) - e^{2xy}(2x+x^2y-xy^2-2y)^2$$

$$D(x, -x) = -x^2e^{-2x^2}(2-x^2-x^2)(-2+x^2+x^2) - e^{-2x^2}(2x-x^3-x^3+2x)^2 =$$

$$= 4x^2e^{-2x^2}\left((1-x^2)^2 - (2-x^2)^2\right) = 4x^2e^{-2x^2}(2x^2-3)_{x=1/\sqrt{2}} = 2e^{-1}(1-3) = -4e^{-1} < 0$$

Thus we got a saddle point.

Ex 9. Find absolute min/max of $f(x,y) = x^2 - xy + y^2$ in rectangular $-1 \leq x, y \leq 1$

The critical points only at zeros of partial derivatives (why?):

$$\begin{cases} f'_x = 2x - y = 0 \\ f'_y = -x + 2y = 0 \end{cases} \Rightarrow 2y = x = y/2 \Rightarrow (x,y) = (0,0)$$

$D = f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 2 - (-1)(-1) = 3 > 0$ and also $f''_{xx} = 2 > 0$ therefore we got a local minimum at $(0,0)$.

Next we look at boundaries we have

$$f(\pm 1, y) = (\pm 1)^2 \pm y + y^2$$

$$f(x, \pm 1) = x^2 + x + (\pm 1)^2$$

which are smiling parabolas (second derivative = 2 > 0) centered (with min) at $\pm \frac{1}{2}$ respectively and a minimum value is $\frac{3}{4}$. This value is a local minimum on the boundary, but is above the minimum inside the domain, so the minimum at $(0,0)$ is an absolute (or global) minimum. For the global maximum we have to look at the values of the function at the corners, thus

$$f(\pm 1, \pm 1) = 1^2 - 1 + 1^2 = 1$$

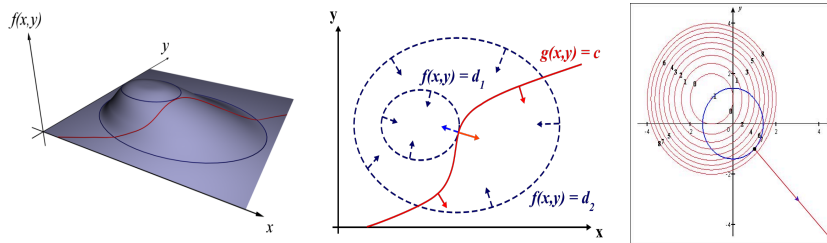
$$f(\pm 1, \mp 1) = 1^2 + 1 + 1^2 = 3$$

Thus, we got two absolute maximum points at $(-1,1)$ and $(1,-1)$.

4.8 Lagrange Multipliers (11.8)

In this section we discuss Max\Min problem with constraint. Examples of such problems are as following:

- 1) The tallest student in 1321 class. We have a function that return height of people and the constraint – students of our class.
- 2) When we looking for max/min of a function in closed bounded set D we have to find min\max on the boundary. Here the constraint here is that we looking for the points along the closed curve which is the boundary of the set D.
- 3) Consider extreme values on a surface S along a curve C.



For convenience we consider 2D case. If we sketch a contour plane of a surface $z=f(x,y)$ together with a constraint $g(x,y)=k$ it can be seen that they are tangent at extreme values (with respect to constraint), because otherwise we can move along the constrain on the surface to get higher\lower values. Thus, geometrically, the function and the constraint should have proportional normal, or in other words, they have proportional gradients.

Alg: To find the maximum/minimum values of $f(x,y)/f(x,y,z)$ subject to constraint $g(x,y)=k/g(x,y,z)=k$ [assuming that these extreme values exists and $\nabla g \neq 0$ on the surface $g=k$].

- 1) Find $(x,y)/(x,y,z)$ such that

$$\begin{aligned} \nabla f(x,y) &= \lambda \nabla g(x,y) & \text{and} & & g(x,y) &= k \\ \nabla f(x,y,z) &= \lambda \nabla g(x,y,z) & \text{and} & & g(x,y,z) &= k \end{aligned}$$
- 2) Evaluate f at points you find and order to find min/max

One can also view this problem as a min/max of the Lagrange function

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda \cdot (g(x,y,z) - k)$$

$$L(x,y,\lambda) = f(x,y) + \lambda \cdot (g(x,y) - k)$$

The number λ is called Lagrange Multiplier.

Ex 1. Find extreme values of $z = x^2 + y^2$ under constrain $x + y = 1$

Solution 1: rewrite the constraint as $y = 1 - x$, then $z = x^2 + (1 - x)^2$ and we can find extreme values using $z' = 2x - 2(1 - x) = 4x - 2$ and $z'' = 4 > 0$, so we got minimum at $(x,y) = (0.5, 0.5)$.

Solution 2, using Lagrange multipliers:

$$\bar{\nabla}z = \lambda \bar{\nabla}(x+y-1)$$

$$\langle 2x, 2y \rangle = \lambda \langle 1, 1 \rangle \quad L = x^2 + y^2 - \lambda(x+y-1)$$

$$\langle x, y \rangle = \lambda \langle 1, 1 \rangle \quad \text{or} \quad \begin{cases} L_x = 2x - \lambda = 0 \\ L_y = 2y - \lambda = 0 \\ L_\lambda = x + y - 1 = 0 \end{cases} \Rightarrow \begin{cases} x + y - \lambda = 0 \\ x + y - 1 = 0 \end{cases} \Rightarrow \lambda = 1 \Rightarrow (x, y) = (0.5, 0.5)$$

$$x = y$$

$$0 = x + y - 1 = 2x - 1$$

$$x = y = 0.5$$

Ex 2. Find absolute min/max of $f(x, y) = x^2 + y^2 + 2x - 2y$ on $x^2 + y^2 \leq 1/4$

In $x^2 + y^2 < 1/4$ we have $\begin{cases} f'_x = 2x + 2 = 0 \\ f'_y = 2y - 2 = 0 \end{cases}$ and so we found an extreme value at $(x, y) = (-1, 1)$,

however this point is outside the domain of interest.

$$\bar{\nabla}f = \lambda \bar{\nabla}g$$

$$\langle 2x + 2, 2y - 2 \rangle = \lambda \langle 2x, 2y \rangle \Rightarrow \langle x + 1, y - 1 \rangle = \lambda \langle x, y \rangle$$

On $x^2 + y^2 = 1/4$ $\begin{cases} x + 1 = \lambda x \\ y - 1 = \lambda y \end{cases} \Rightarrow \begin{cases} (\lambda - 1)x = 1 \\ (\lambda - 1)y = -1 \end{cases} \Rightarrow x = -y$

$$x^2 + y^2 = x^2 + (-x)^2 = 2x^2 = 1/4 \Rightarrow x = \pm\sqrt{1/8}$$

Therefore the extreme values at $(x, y) = (\sqrt{1/8}, -\sqrt{1/8}), (-\sqrt{1/8}, \sqrt{1/8})$

$$f(x, -x) = x^2 + x^2 + 2x + 2x = 2x^2 + 4x \Big|_{x=\pm\sqrt{1/8}} = \begin{cases} 1/4 + 4/\sqrt{8} & \text{max} \\ 1/4 - 4/\sqrt{8} & \text{min} \end{cases} = \begin{cases} 1/4 + 2/\sqrt{2} \\ 1/4 - 2/\sqrt{2} \end{cases}$$

Ex 3. Let a, b, c be lengths of edges of a triangle. Maximize the function

$$F = a^2 + b^2 + c^2 \text{ under constraint } g(b, c, \theta) = \frac{1}{2}bc \sin \theta = S.$$

Solution: Since $a^2 = b^2 + c^2 - 2bc \cos \theta$ we rewrite

$$F(a, b, c) = a^2 + b^2 + c^2 = 2b^2 + 2c^2 - 2bc \cos \theta = f(b, c, \theta)$$

Next we use Lagrange multipliers

$$\bar{\nabla}f = \lambda \bar{\nabla}g$$

$$\langle 4b - 2c \cos \theta, 4c - 2b \cos \theta, 2bc \sin \theta \rangle = \lambda \frac{1}{2} \langle c \sin \theta, b \sin \theta, bc \cos \theta \rangle$$

$$2bc \sin \theta = \lambda \frac{1}{2} bc \cos \theta \Rightarrow \lambda = 4 \tan \theta$$

$$\begin{cases} 4b - 2c \cos \theta = 4 \tan \theta \cdot \frac{1}{2} c \sin \theta \\ 4c - 2b \cos \theta = 4 \tan \theta \cdot \frac{1}{2} b \sin \theta \end{cases} \Rightarrow \begin{cases} 2b = c(\tan \theta \sin \theta + \cos \theta) \\ 2c = b(\tan \theta \sin \theta + \cos \theta) \end{cases} \Rightarrow c = b$$

$$\Rightarrow 2 = \tan \theta \sin \theta + \cos \theta = \frac{\sin^2 \theta}{\cos \theta} + \cos \theta = \frac{1}{\cos \theta}$$

$$a^2 = b^2 + c^2 - 2bc \cos \theta = 2b^2 - 2b^2 \cos \theta = 2b^2(1 - \cos \theta) = 2b^2(1 - \frac{1}{2}) = b^2 \Rightarrow a = b$$

Finally, we got Equilateral triangle.