4.5 The Chain Rule (11.5)

Recall: Chain Rule for y=f(x), x=g(t): $\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$

Thm: Chain Rule for z=f(x,y), x(t)=g(t), y(t)=h(t): $\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$

$$\frac{dz}{dt}=\frac{d}{dt}f\left(g(t),h(t)\right)=\lim_{\delta\to 0}\frac{f\left(g(t+\delta),h(t+\delta)\right)-f\left(g(t),h(t)\right)}{\delta}=$$

$$=\lim_{\delta\to 0}\frac{f\left(g(t)+\Delta^{\delta}g,h(t)+\Delta^{\delta}h\right)-f\left(g(t),h(t)\right)}{\delta}=$$

$$=\lim_{\delta\to 0}\frac{\cancel{f\left(g(t),h(t)\right)}+f_x\left(g(t),h(t)\right)\Delta^{\delta}g+f_y\left(g(t),h(t)\right)\Delta^{\delta}h-\cancel{f\left(g(t),h(t)\right)}}{\delta}$$

$$=f_x\left(g(t),h(t)\right)\lim_{\delta\to 0}\frac{\Delta^{\delta}g}{\delta}+f_y\left(g(t),h(t)\right)\lim_{\delta\to 0}\frac{\Delta^{\delta}h}{\delta}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

Ex 1. Find $\frac{dz}{dt}$ for $z=3xy^2-4x^3y, x=e^t, y=\sin t$

$$\frac{dz}{dt}=\left(3y^2-12x^2y\right)e^t+\left(6xy-4x^3\right)\cos t=\left(3\sin^2 t-12e^{2t}\sin t\right)e^t+\left(6e^t\sin t-4e^{3t}\right)\cos t$$

Ex 2. Let $z_x=-z_y$. Find y(x) such that $u(x)=z\left(x,y(x)\right)$ is constant.

The derivative of a constant function is equal zero, therefore

$$u'(x)=\frac{dz}{dx}=\frac{\partial z}{\partial x}\frac{dx}{dx}+\frac{\partial z}{\partial y}\frac{dy}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{dy}{dx}=z_x+z_y\frac{dy}{dx}=-z_y+z_y\frac{dy}{dx}=z_y\left(\frac{dy}{dx}-1\right)=0$$

Thus either one is true $z_y=0$ or $\frac{dy}{dx}=1$.

If $z_y=0$ then $z_x=-z_y=0$ and therefore, *z* is constant function regardless of y(x).

Otherwise, we got $\frac{dy}{dx}=1\Rightarrow y(x)=\int\frac{dy}{dx}dx=\int 1\cdot dx=x+C$.

Thm: Chain Rule for z=f(x,y), x(t)=g(s,t), y(t)=h(s,t):

$$\frac{\partial z}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\qquad\qquad\frac{\partial z}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Ex 3. $z=xy+\sin(xy)\ x=\cos st, y=\sin st$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=-(y+y\cos xy)t\sin st+(x+x\cos xy)t\cos st=$$

$$=-(1+\cos(\cos st\sin st))t\sin^2 st+(1+\cos(\cos st\sin st))t\cos^2 st$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}=-(y+y\cos xy)s\sin st+(x+x\cos xy)s\cos st=$$

$$=-(1+\cos(\cos st\sin st))s\sin^2 st+(1+\cos(\cos st\sin st))s\cos^2 st$$

Ex 4. $z = f(x,y)\ x = r\cos\theta, y = r\sin\theta$

$$x = r\cos\theta, y = r\sin\theta \Rightarrow r(x,y) = \sqrt{x^2+y^2}; \theta(x,y) = \arctan\frac{y}{x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r}\frac{x}{\sqrt{x^2+y^2}} - \frac{\partial z}{\partial \theta}\frac{y}{x^2+y^2} = \frac{\partial z}{\partial r}\frac{r\cos\theta}{r} - \frac{\partial z}{\partial \theta}\frac{r\sin\theta}{r^2} = \frac{\partial z}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial z}{\partial \theta}\sin\theta$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial z}{\partial r}\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial z}{\partial \theta}\frac{x}{x^2+y^2} = \frac{\partial z}{\partial r}\frac{r\sin\theta}{r} + \frac{\partial z}{\partial \theta}\frac{r\cos\theta}{r^2} = \frac{\partial z}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial z}{\partial \theta}\cos\theta$$

Ex 5. $z = f(x,y), x = s - t, y = s + t$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = -\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$$

$$(z_x)^2 - (z_y)^2 = (z_x - z_y)(z_x + z_y) = z_s z_t$$

Thm: Implicit differentiation

1) Given equation $F(x, y(x)) = 0$, one differentiate both sides with respect to x to get $F_y \frac{\partial y}{\partial x} + F_x \underbrace{\frac{\partial x}{\partial x}}_{=1} = 0$ then solve it for $\frac{dy}{dx} = -\frac{F_x}{F_y}$

2) Similarly, given $F(x, y, z(x, y)) = 0$:
   a. Differentiating with respect to x gives $F_x \underbrace{\frac{\partial x}{\partial x}}_{=1} + F_y \underbrace{\frac{\partial y}{\partial x}}_{=0} + F_z \frac{\partial z}{\partial x} = 0 \Rightarrow z_x = -\frac{F_x}{F_z}$
   b. Differentiating with respect to y gives $F_x \underbrace{\frac{\partial x}{\partial y}}_{=0} + F_y \underbrace{\frac{\partial y}{\partial y}}_{=1} + F_z \frac{\partial z}{\partial y} = 0 \Rightarrow z_y = -\frac{F_y}{F_z}$

Ex 6. $x + \cos xy + \sin xz = 0$

$$1 - y\sin xy + z\cos xz + z_x \cos xz = 0 \Rightarrow z_x = -\frac{1 - y\sin xy + z\cos xz}{x\cos xz} = -\frac{F_x}{F_z}$$

$$-x\sin xy + xz_y \cos xz = 0 \Rightarrow z_y = \frac{x\sin xy}{x\cos xz} = -\frac{F_y}{F_z}$$

4.6 Directional Derivatives and the Gradient Vector (11.6)

The derivative of *f*(*x,y,z*) with respect to *x, y* or *z* are also called the derivative in the direction of *x, y, z*. We can generalize the idea and to find the derivative of the function in direction of any line $(x,y)=(at+x_0,bt+y_0)$.

Def: The (Directional) Derivative of a function f (x,y) in a direction of a unit vector $\vec{u}=\langle a,b\rangle$ is given by $D_{\vec{u}}f(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+ah,y_0+bh)a-f(x_0,y_0)}{h}$ if the limit exists.

Thm: $D_{\vec{u}}f(x_0,y_0)=f_x(x_0,y_0)a+f_y(x_0,y_0)b$

Proof: Define $g(t)=f(at+x_0,bt+y_0)$. Note that $g'(t)=f_x(at+x_0,bt+y_0)a+f_y(at+x_0,bt+y_0)b$ at $t=0$ gives $g'(0)=D_{\vec{u}}f(x_0,y_0)$ therefore $D_{\vec{u}}f(x_0,y_0)=g'(0)=f_x(x_0,y_0)a+f_y(x_0,y_0)b$.

Def: Gradient vector of a function of 2 variables is defined as

$$\overrightarrow{\nabla f}(x,y)=\langle f_x(x,y),f_y(x,y)\rangle=\tfrac{\partial f}{\partial x}\,\mathrm{i}+\tfrac{\partial f}{\partial y}\,\mathrm{j}$$

Similarly $\overrightarrow{\nabla f}(x,y,z)=\langle f_x(x,y,z),f_y(x,y,z),f_z(x,y,z)\rangle=\tfrac{\partial f}{\partial x}\,\mathrm{i}+\tfrac{\partial f}{\partial y}\,\mathrm{j}+\tfrac{\partial f}{\partial z}\,\mathrm{k}$

And $\overrightarrow{\nabla f}(x_1,\ldots,x_n)=\langle f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\rangle$

Def: Another definition of directional derivative:

$$D_{\langle a,b\rangle}f(x,y)=\overrightarrow{\nabla f}(x,y)\cdot\langle a,b\rangle$$

$$D_{\langle a,b,c\rangle}f(x,y,z)=\overrightarrow{\nabla f}(x,y,z)\cdot\langle a,b,c\rangle$$

$$D_{\langle y_1,\ldots,y_n\rangle}f(x_1,\ldots,x_n)=\overrightarrow{\nabla f}(x_1,\ldots,x_n)\cdot\langle y_1,\ldots,y_n\rangle$$

Def: Yet another way to define directional derivative: let $f:\mathbb{R}^n\to\mathbb{R}f$ be a real valued function with a vector space domain $\mathbb{R}^n$, i.e. $f(\vec{x})\in\mathbb{R}$. Thus, the directional derivative is given by

$$D_{\vec{u}}f(\vec{x})=\lim_{h\to 0}\frac{f(\vec{x}+h\vec{u})-f(\vec{x})}{h}$$

Ex 7. Let $f(x,y,z)=xy\cos z$ and let $\vec{u}=\langle 1,2,3\rangle$:

$$D_{\vec{u}}f(1,2,0)=\langle y\cos z,x\cos z,-xy\sin z\rangle_{(x,y,z)=(1,2,0)}\cdot\langle 1,2,3\rangle=\langle 2,1,-2\rangle\cdot\langle 1,2,3\rangle=2+2-6=-2$$

Thm: The maximum of a directional derivative at given point occurs when the vector u has the same direction as the gradient vector.

Proof: $D_{\vec{u}}f=\overrightarrow{\nabla f}\cdot\vec{u}=|\overrightarrow{\nabla f}||\vec{u}|\cos\theta$, where θ is the angle between $\overrightarrow{\nabla f}$ and $|\vec{u}|$. Since $\cos\theta$ reach maximum has the maximum 1, which occurs at $\theta=0$, in other words when the vector u has the same direction as the gradient vector.