4.4 Tangent Planes and Linear Approximations (11.4)

In Calculus 1 we approximated function using linear approximation $f(x+h)=f(x)+hf'(x)$. We want to develop similar technique for function of 2 variables.

A plane $\vec{n}(r-r_0)=\tilde{A}(x-x_0)+\tilde{B}(y-y_0)+\tilde{C}(z-z_0)=0$ can be described as $z-z_0=A(x-x_0)+B(y-y_0)$, where $A=\tilde{A}/\tilde{C},\ B=\tilde{B}/\tilde{C}$

Def: Let S be surface described by a function z=f(x,y) and suppose that *f* has continuous partial derivatives. An equation of the tangent plane to S at point $P(x_0,y_0,z_0)$ is given by:

$$z-z_0=f_x(x-x_0)+f_y(y-y_0)$$

Similarly a surface defined by f(x,y,z)=0 the tangent plane is $f_x(x-x_0)+f_y(y-y_0)+f_z(z-z_0)=0$

Def: A tangent plane to parametric surface $\vec{r}(u,v)=x(u,v)\mathrm{i}+y(u,v)\mathrm{j}+z(u,v)\mathrm{k}$ is given using a normal vector $\vec{n}=\vec{r}_u\times\vec{r}_v=\langle x_u,y_u,z_u\rangle\times\langle x_v,y_v,z_v\rangle$.

Ex 1. Find a tangent plane to $z=3x^3+4y^2$ at point $(1,1,7)$

$$f_x=9x^2+4y^2\Big|_{(x,y)=(1,1)}=13$$

$$f_y=3x^3+8y\Big|_{(x,y)=(1,1)}=11$$

$$z-7=13(x-1)+11(y-1)=13x+11y-24\Rightarrow z=13x+11y-17$$

Ex 2. Find a tangent plane to $z=\cos(xy)$ at point $\left(1,\frac{\pi}{3},\frac{1}{2}\right)$

$$z-\frac{1}{2}=\frac{\partial}{\partial x}f\left(1,\frac{\pi}{3}\right)(x-1)+\frac{\partial}{\partial y}f\left(1,\frac{\pi}{3}\right)\left(y-\frac{\pi}{3}\right)$$

$$\frac{\partial f}{\partial x}=-y\sin xy;\ \frac{\partial f}{\partial y}=-x\sin xy$$

$$z-\frac{1}{2}=-\frac{\pi\sqrt{3}}{6}(x-1)-\frac{\sqrt{3}}{2}\left(y-\frac{\pi}{3}\right)\Leftrightarrow\frac{\pi\sqrt{3}}{6}(x-1)+\frac{\sqrt{3}}{2}\left(y-\frac{\pi}{3}\right)+z-\frac{1}{2}=0$$

Ex 3. Find a tangent plane to $\cos xyz+z=0$

$$\frac{\partial F}{\partial x}\left(\frac{\pi}{6},1,1\right)\left(x-\frac{\pi}{6}\right)+\frac{\partial F}{\partial y}\left(\frac{\pi}{6},1,1\right)(y-1)+\frac{\partial F}{\partial z}\left(\frac{\pi}{6},1,1\right)(z-1)=0$$

$$\frac{\partial F}{\partial x}=-yz\sin xyz\qquad\frac{\partial F}{\partial y}=-xz\sin xyz\qquad\frac{\partial F}{\partial x}=-xy\sin xyz;$$

$$\frac{\partial F}{\partial x}\left(\frac{\pi}{6},1,1\right)=-\frac{1}{2}\qquad\frac{\partial F}{\partial x}\left(\frac{\pi}{6},1,1\right)=-\frac{\pi}{12}\qquad\frac{\partial F}{\partial x}\left(\frac{\pi}{6},1,1\right)=-\frac{1}{2}\Rightarrow$$

$$-\frac{1}{2}\left(x-\frac{\pi}{6}\right)-\frac{\pi}{12}(y-1)-\frac{1}{2}(z-1)=0\Leftrightarrow$$

$$6\left(x-\frac{\pi}{6}\right)+\pi(y-1)+6(z-1)=0\Leftrightarrow 6x+\pi y+6z-2\pi+6=0$$

Ex 4. Find a plane tangent to $\vec{r}(x,y)=\langle xy, y\cos x, x\cos y\rangle$ at $(0,\pi)$

$$\vec{r}_x=\langle y,-y\sin x,\cos y\rangle$$
$$\vec{r}_y=\langle x,\cos x,-x\sin y\rangle$$
$$\vec{n}=\vec{r}_x\times\vec{r}_y=\langle -\cos x\cos y+xy\sin x\sin y, x\cos y+xy\sin y, y\cos x+xy\sin x\rangle_{(x,y)=(0,0)}=\langle 1,0,\pi\rangle$$
$$(x-x_0)+(y-y_0)+\pi(z-z_0)=0$$

Def: Suppose that a function *f*(*x,y*) has continuous partial derivatives. A function $L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$ is called **linearization of *f*.**

A **linear approximation** or a **tangent plane approximation** of *f* at (a,b) is defined by

$$f(x,y)\approx f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)=L(x,y) \text{ or equivalently}$$
$$f(a+h_x,b+h_y)\approx f(a,b)+f_x(a,b)h_x+f_y(a,b)h_y$$

We defined linear approximation for function with continuous partial derivatives, however what happens when the derivatives aren't continuous?

Ex 5. Find linear approximation of $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$ near 0.

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{\Delta x\cdot 0}{\Delta x^2+0^2}-0\right)=0$$
$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{1}{\Delta y}\left(\frac{0\cdot\Delta y}{0^2+\Delta y^2}-0\right)=0$$

therefore: $f(x,y)\approx 0$ however $f(x,x)=\dfrac{x^2}{x^2+x^2}=\dfrac{1}{2}$, which mean the approximation is bad.

Def: The function z=f(x,y) is **differentiable** at (a,b) if $\Delta z=\Delta f=f(a+\Delta x,b+\Delta y)-f(a,b)$ can be expressed in the form

$$\Delta z=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\varepsilon_1(\Delta x,\Delta y)\Delta x+\varepsilon_2(\Delta x,\Delta y)\Delta y$$

where $\lim_{(\Delta x,\Delta y)\to 0}\varepsilon_1(\Delta x,\Delta y)=0$ and $\lim_{(\Delta x,\Delta y)\to 0}\varepsilon_2(\Delta x,\Delta y)=0$

or equivalently $\Delta z=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\varepsilon(\Delta x,\Delta y)\sqrt{\Delta x^2+\Delta y^2}$ where $\lim_{(\Delta x,\Delta y)\to 0}\varepsilon(\Delta x,\Delta y)=0$.

Where the second form is more convenient for calculations, since one solves it for

$$\varepsilon(\Delta x,\Delta y)=\frac{f(a+\Delta x,b+\Delta y)-f(a,b)-f_x(a,b)\Delta x-f_y(a,b)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}$$

Ex 6. Verify differentiability of $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{2x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$ at $(0,0)$

We first verify it is continuous at 0 by rewriting it into polar coordinates:

$$\frac{x^3+y^3}{2x^2+y^2}=\frac{(r\cos\theta)^3+(r\sin\theta)^3}{2(r\cos\theta)^2+(r\cos\theta)^2}=\frac{r^3}{r^2}\frac{\cos^3\theta+\sin^3\theta}{2\cos^2\theta+\sin^2\theta}=r\frac{\cos^3\theta+\sin^3\theta}{2\cos^2\theta+\sin^2\theta}=r\frac{\cos^3\theta+\sin^3\theta}{\cos\theta+1}$$

Therefore $\lim\limits_{(x,y)\to 0}\dfrac{x^3+y^3}{2x^2+y^2}=\dfrac{\cos^3\theta+\sin^3\theta}{\cos\theta+1}\lim\limits_{r\to 0}r=0$, thus it is continuous.

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{\Delta x^3+0^3}{2\Delta x^2+0^2}-0\right)=\lim_{\Delta x\to 0}\frac{\Delta x^3}{2\Delta x^3}=\frac{1}{2}$$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{1}{\Delta y}\left(\frac{0^3+\Delta y^3}{2\cdot 0^2+\Delta y^2}-0\right)=\lim_{\Delta y\to 0}\frac{\Delta y^3}{\Delta y^3}=1$$

$$\varepsilon(\Delta x,\Delta y)=\frac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}=\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}\left[\frac{\Delta x^3+\Delta y^3}{2\Delta x^2+\Delta y^2}-\frac{\Delta x}{2}-\Delta y\right]$$

The function isn't differentiable at (0,0) since (it is enough to see it is not 0 on any curve)

$$\lim_{(\Delta x,\Delta y)\to 0}\varepsilon(\Delta x,\Delta y)\underset{\text{if exists}}{=}\lim_{(\Delta x,\Delta x)\to 0}\frac{1}{\sqrt{2\Delta x^2}}\left[\frac{2\Delta x^3}{3\Delta x^2}-\frac{\Delta x}{2}-\Delta x\right]=\lim_{(\Delta x,\Delta x)\to 0}\frac{\Delta x}{\sqrt{2\Delta x^2}}\left[\frac{2}{3}-\frac{3}{2}\right]\neq 0$$

Ex 7. Verify differentiability of $f(x,y)=\begin{cases}e^{-1/(x^2+y^2)} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$ at 0

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{e^{-1/\Delta x^2}-0}{\Delta x}=$$

$$=\lim_{\Delta x=h\to 0}\frac{1/h}{e^{1/h^2}}\underset{\substack{L.H.R.\\ "\infty/\infty"}}{=}\lim_{h\to 0}\frac{-1/h^2}{-(2/h^3)e^{1/h^2}}=\lim_{h\to 0}\frac{h}{2e^{1/h^2}}=\lim_{h\to 0}h\cdot\frac{1}{2e^{1/h^2}}=0$$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0-e^{-1/\Delta y^2}}{\Delta y}=0$$

$$\varepsilon(\Delta x,\Delta y)=\frac{f(0+\Delta x,0+\Delta y)-f(0,0)+f_x(0,0)\Delta x+f_y(0,0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}=\frac{e^{-\frac{1}{\Delta x^2+\Delta y^2}}}{\sqrt{\Delta x^2+\Delta y^2}}=\frac{e^{-1/r^2}}{r}\to 0$$

Thus $\lim\limits_{(\Delta x,\Delta y)\to 0}\varepsilon(\Delta x,\Delta y)=0$ and the function is differential everywhere

Thm: If the partial derivatives of *f* exists near (a,b) and continuous at (a,b) then f is differentiable.

Def: An analogy to the differentials in 1D $dy=f'(x)dx$ is called a **total differential** in 2D and 3D and is given by $df=f_x(x,y)dx+f_y(x,y)dy$ and $df=f_x(x,y,z)dx+f_y(x,y,z)dy+f_z(x,y,z)dz$ respectively.