4.3 Partial Derivatives (11.3)

Def: A partial derivative of a function of several variables is a derivative with respect of one of those variables why the other variables consider constant. For a function $f(x,y)$, the partial derivatives are:

$$D_x f = \frac{\partial}{\partial x} f(x,y) = f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$
$$D_y f = \frac{\partial}{\partial y} f(x,y) = f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

For a function $f(x,y,z)$, the partial derivatives are:

$$D_x f = \frac{\partial}{\partial x} f(x,y,z) = f_x(x,y,z) = \lim_{h\to 0}\frac{f(x+h,y,z)-f(x,y,z)}{h}$$
$$D_y f = \frac{\partial}{\partial y} f(x,y,z) = f_y(x,y,z) = \lim_{h\to 0}\frac{f(x,y+h,z)-f(x,y,z)}{h}$$
$$D_z f = \frac{\partial}{\partial z} f(x,y,z) = f_z(x,y,z) = \lim_{h\to 0}\frac{f(x,y,z+h)-f(x,y,z)}{h}$$

Similarly, for a function $f(x_1,x_2,...x_n)$ the partial derivatives $D_1 f = f_1, D_2 f = f_2, ..., D_n f = f_n$

Ex 1. Find partial derivatives of $f(x,y) = \frac{\cos(xy)}{x^2+y^2}$

$$f'_x = -\frac{\sin(xy)yx^2+\sin(xy)y^3+2\cos(xy)x}{(x^2+y^2)^2} \qquad f'_y = -\frac{\sin(xy)xy^2+\sin(xy)x^3+2\cos(xy)y}{(x^2+y^2)^2}$$

Ex 2. Verify whether the partial derivatives of

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases} \text{ continuous or not.}$$

The function $f(x,y)$ is continuous at every $(x,y)\neq(0,0)$ as a fractional function, it is also continuous at $(x,y)=(0,0)$ since

$$\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)}\frac{r^2\cos\theta\sin\theta}{r} = \lim_{(x,y)\to(0,0)} r\cos\theta\sin\theta = 0 = f(0,0)$$

Next we find partial derivatives at $(x,y)=(0,0)$

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h\cdot 0/\sqrt{h^2}}{h} = 0 \text{ and } f_y(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0\cdot h/\sqrt{h^2}}{h} = 0$$

For $(x,y)\neq(0,0)$:

$$f_x(x,y)=\frac{y\sqrt{x^2+y^2}-xy\frac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2}=\frac{\frac{x^2y+y^3-x^2y}{\sqrt{x^2+y^2}}}{x^2+y^2}=\frac{y^3}{\left(x^2+y^2\right)^{3/2}}$$

$$f_y(x,y)=\frac{x\sqrt{x^2+y^2}-xy\frac{2y}{2\sqrt{x^2+y^2}}}{x^2+y^2}=\frac{\frac{x^3+xy^2-y^2x}{\sqrt{x^2+y^2}}}{x^2+y^2}=\frac{x^3}{\left(x^2+y^2\right)^{3/2}}$$

Finally, f_x, f_y aren't continuous at $(x,y)=(0,0)$ since

$$\lim_{(x,y)\to(0,0)} f_x(x,y)=\lim_{(x,x)\to(0,0)} f_x(x,x)=\lim_{x\to 0}\frac{x^3}{\left(2x^2\right)^{3/2}}=\frac{1}{2^{3/2}}\neq 0=f_x(0,0)$$

$$\lim_{(x,y)\to(0,0)} f_y(x,y)=\lim_{(x,x)\to(0,0)} f_y(x,x)=\lim_{x\to 0}\frac{x^3}{\left(2x^2\right)^{3/2}}=\frac{1}{2^{3/2}}\neq 0=f_y(0,0)$$

4.3.1 Higher derivatives

Partial derivatives of a function of several variables are functions of several variables, and therefore, may have their partial derivatives. We denote it as:

$$f_{xx}=\frac{\partial f}{\partial x^2}\qquad f_{xy}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial f}{\partial y\partial x}\qquad f_{yx}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial f}{\partial x\partial y}\qquad f_{yy}=\frac{\partial f}{\partial y^2}$$

Thm: If f_{xy}, f_{yx} continuous around (a,b) then $f_{xy},(a,b)=f_{yx}(a,b)$

Ex 3. Let $f(x,y)=\frac{y^2}{x^2}+y+\frac{x^2+y^2}{2}$. Find $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$

$$f_x(x,y)=-2\frac{y^2}{x^3}+x \qquad f_y(x,y)=\frac{2y}{x^2}+1+y$$

$$f_{xx}(x,y)=6\frac{y^2}{x^4}+1 \qquad f_{yy}(x,y)=\frac{2}{x^2}+1$$

$$f_{xy}(x,y)=-4\frac{y}{x^3} \qquad f_{yx}(x,y)=-4\frac{y}{x^3}$$

4.3.2 Partial differential equations

Def: Partial differential equations are differential equations that involve partial derivatives.

Ex 4. Find k such that $u=\sin ax\sin by$ satisfies Helmholtz equation $u_{xx}+u_{yy}+k^2u=0$.

$$u_x=a\cos ax\sin by \qquad u_{xx}=-a^2\sin ax\sin by=-a^2u$$

$$u_y=b\sin ax\cos by \qquad u_{yy}=-b^2\sin ax\sin by=-b^2u$$

$$u_{xx}+u_{yy}=-\left(a^2+b^2\right)u\Rightarrow k^2=a^2+b^2$$