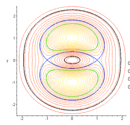
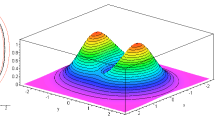
4 Partial Derivatives

4.1 Functions of Several Variables (11.1)

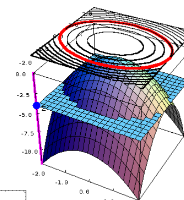
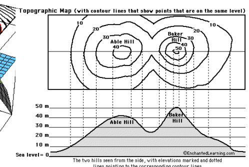
We already met function of two variables in sections 9.6 and 10.5, in this chapter we will learn more about functions of 2 and 3 variables and talk a (very) little about functions of more variables. We already discussed examples of such functions, some more examples can be found in the book.

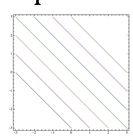
Visualization of functions of 2 variables

Def: The **level curves\contour lines** of a function *f* of 2 variables are the curves with equations *f*(*x*,*y*)=*k*, where *k* is a constant in the range of *f*.

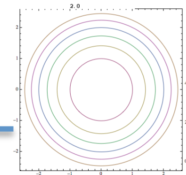
One can draw a surface in 2D using level curves as projections of the curves onto xy plane, often by using equally spaced set of constants *k*. The well-known example is the topographical maps.

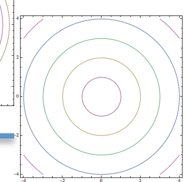
Ex 7. Draw a contour plot of the plane z=x+y



Ex 8. Draw a contour plot of the surface $z = x^2 + y^2$

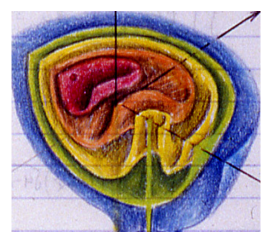


Ex 9. Draw a contour plot of the surface $z^2 = x^2 + y^2$



Note: If *f*(*x*,*y*) is continuous (not defined yet), then the level curves at different heights are never cross.

Functions of 3 and more variables



Def: The level surfaces of a function *f* of 3 variables are the surfaces with equations *f*(*x*,*y*,*z*)=*k* where *k* is a constant in the range of *f*.

One can use level surfaces to visualize functions of 3 variables, but there is no way to describe the real image, because the image is in 4th dimensions.

Def: A function multiple variables can be viewed in several ways, each one may be useful in different situations; we may meet it in future.

1. A function of n real variables $f(x_1, x_2, \dots, x_n)$
2. A vector function $f(\vec{x}) = f(\langle x_1, x_2, \dots, x_n \rangle)$, where $\vec{x} = \langle x_1, x_2, \dots, x_n \rangle$
3. A single point variable function $f((x_1, x_2, \dots, x_n)) = f(x_1, x_2, \dots, x_n)$, where the n-tuple $(x_1, x_2, \dots, x_n)$ is considered a point on n-dimensional space.

4.2 Limits and Continuity (11.2)

Def: We write $\lim\limits_{(x,y)\to(a,b)} f(x,y)=L$ and say the limit of f(x,y) as (x,y) approaches (a,b) is L if we can make the values of f(x,y) as close to L as we like by taking the point (x,y) sufficiently colose to the point (a,b), but not equal to (a,b).

Thm: If the limit $\lim\limits_{(x,y)\to(a,b)} f(x,y)=L$ exists then for any curve $(x,y)=(x(t),y(t))$ the function $\tilde{f}(t)=f(x(t),y(t))$ has the limit at t_0 s.t. $(a,b)=(x(t_0),y(t_0))$ and $\lim\limits_{t\to t_0}\tilde{f}(t)=L$.

Ex 1. $$\lim_{(x,y)\to(2,3)}\frac{x^2+3y^2+2xy+5}{5xy+x^2-y^2-3}=\frac{\lim_{(x,y)\to(2,3)} x^2+3y^2+2xy+5}{\lim_{(x,y)\to(2,3)} 5xy+x^2-y^2-3}=\frac{2^2+3\cdot 3^2+2\cdot 2\cdot 3+5}{5\cdot 2\cdot 3+2^2-3^2-3}=\frac{48}{24}=2$$

Note: If we find 2 different curves $(x(t),y(t))$ and $(\tilde{x}(t),\tilde{y}(t))$ for which $\lim\limits_{(x(t),y(t))\to(a,b)} f(x(t),y(t))=L_1$ and $\lim\limits_{(\tilde{x}(t),\tilde{y}(t))\to(a,b)} f(\tilde{x}(t),\tilde{y}(t))=L_2$, such that $L_1\neq L_2$ then the limit $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ doesn't exists.

Ex 2. If the limit $\lim\limits_{(x,y)\to 0}\frac{x+y^2}{\sqrt{x^2+y^2}}$ exists, then it the same as on curve x=x:

$$\lim_{(x,x)\to 0}\frac{x+x^2}{\sqrt{x^2+x^2}}=\lim_{(x,x)\to 0}\frac{x}{|x|}\frac{1+x}{\sqrt{2}}=\frac{1}{\sqrt{2}}\lim_{(x,x)\to 0^\pm}\frac{x}{|x|}(1+x)=\pm 1$$ thus, the one sided limits are ± 1, and therefore the limit doesn't exists.

Ex 3. $$\lim_{(x,y)\to 0}\frac{xy^2}{x^2+y^4}=\lim_{(x,kx)\to 0}\frac{x^3k^2}{x^2+k^4x^4}=\lim_{(x,kx)\to 0}\frac{xk^2}{1+k^4x^2}=0\neq\frac{1}{2}=\lim_{(y^2,y)\to 0}\frac{y^4}{y^4+y^4}=\lim_{(x,y)\to 0}\frac{xy^2}{x^2+y^4}$$

Ex 4. $$\left|\frac{\sin x^3}{x^2+y^2}\right|\le\left|\frac{\sin x^3}{x^2}\right|=|x|\left|\frac{\sin x^3}{x^3}\right|\to 0\cdot 1$$ and so also

$$\lim_{(x,y)\to 0}\frac{\sin(x^3+y^3)}{x^2+y^2}=\lim_{(x,y)\to 0}\frac{\sin x^3\cos y^3+\sin y^3\cos x^3}{x^2+y^2}=\lim_{(x,y)\to 0}\frac{\sin x^3}{x^2+y^2}\lim_{(x,y)\to 0}\cos y^3+$$
$$+\lim_{(x,y)\to 0}\frac{\sin y^3}{x^2+y^2}\lim_{(x,y)\to 0}\cos x^3=\lim_{(x,y)\to 0}\frac{\sin x^3}{x^2+y^2}+\lim_{(x,y)\to 0}\frac{\sin y^3}{x^2+y^2}=0$$

Ex 5.
$$\lim_{(x,y)\to 0}\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}=\lim_{(x,y)\to 0}\frac{x^2y^2+1-1}{(x^2+y^2)\underbrace{(\sqrt{x^2y^2+1}+1)}_{\neq 0}}=\lim_{(x,y)\to 0}\frac{x^2y^2}{(x^2+y^2)(\sqrt{x^2y^2+1}+1)}=$$
$$=\lim_{r\to 0}\frac{\cancel{r^2}\cos^2\theta\cdot r^2\sin^2\theta}{\cancel{r^2}(\sqrt{r^4\cos^2\theta\sin^2\theta+1}+1)}=\lim_{r\to 0}\frac{\cancel{r^2}\cos^2\theta\sin^2\theta}{\cancel{r^2}\sqrt{\cos^2\theta\sin^2\theta+1/r^4}+1/r^2}=0$$

Ex 6. $\lim\limits_{(x,y)\to 0} \dfrac{x^2 - x\,y}{\sqrt{x} - \sqrt{y}} = \lim\limits_{(x,y)\to 0} \dfrac{x(x- y)}{\sqrt{x} - \sqrt{y}} = \lim\limits_{(x,y)\to 0} \dfrac{x\left(\sqrt{x} + \sqrt{y}\right)\left(\sqrt{x} - \sqrt{y}\right)}{\sqrt{x} - \sqrt{y}} = \lim\limits_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) = 0$

Def: A function *f* of two variables is called **continues at point (a,b)** if $\lim\limits_{(x,y)\to(a,b)} f\left(x,y\right) = f\left(a,b\right)$.

Def: A function *f* of two variables is called **continues** if *f* is continuous at every $\left(a,b\right) \in D$, where *D* is the domain of *f*.

Ex 7. Verify whether there is exist or not a number *c* such that the function $f(x,y) = \begin{cases} \dfrac{x}{y}\sin y\ , & y \neq 0 \\ c & , \ y = 0 \end{cases}$ is continuous.

Such c doesn't exists since:

$$c = \lim_{\substack{(x,y)\to(x_0,0) \\ on\ x=x_0}} f(x,y) = x_0 \lim_{t\to 0}\frac{\sin t}{t} = x_0 \neq x_1 = x_1 \lim_{t\to 0}\frac{\sin t}{t} = \lim_{\substack{(x,y)\to(x_1,0) \\ on\ x=x_1}} f(x,y) = c$$