3.4 Motion in Space: Velocity and Acceleration (10.4)

Def: Let t represent a time and $\vec{r}(t)$ a trajectory of a moving particle. Then the velocity vector is defined by $\vec{v}(t)=\vec{r}\,'(t)$, the speed is given by $v=|\vec{v}(t)|=|\vec{r}\,'(t)|$ and the acceleration reads by $\vec{a}=|\vec{v}\,'(t)|=|\vec{r}\,''(t)|$.

Thm: The acceleration of a particle following the curve $\vec{r}(t)$ consist of 2 components a change of speed $a_T=\frac{d}{dt}|\vec{v}(t)|$ and a change of velocity direction $a_N=\kappa|\vec{v}(t)|^2$, so that $\vec{a}=a_T\vec{T}(t)+a_N\vec{N}(t)$.

Proof:

$$\vec{T}(t)=\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}=\frac{\vec{v}(t)}{|\vec{v}(t)|}\Rightarrow\vec{v}(t)=|\vec{v}(t)|\vec{T}(t)$$

$$\kappa=\frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}=\frac{|\vec{T}\,'(t)|}{|\vec{v}(t)|}\Rightarrow|\vec{T}\,'(t)|=\kappa|\vec{v}(t)|$$

$$\vec{N}(t)=\frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}\Rightarrow\vec{T}\,'(t)=\vec{N}(t)|\vec{T}\,'(t)|=\kappa|\vec{v}(t)|\vec{N}(t)$$

$$\vec{a}(t)=\vec{v}\,'(t)=\left(\frac{d}{dt}|\vec{v}(t)|\right)\vec{T}(t)+|\vec{v}(t)|\vec{T}\,'(t)=\left(\frac{d}{dt}|\vec{v}(t)|\right)\vec{T}(t)+\kappa|\vec{v}(t)|^2\vec{N}(t)$$

3.5 Parametric Surfaces (10.5)

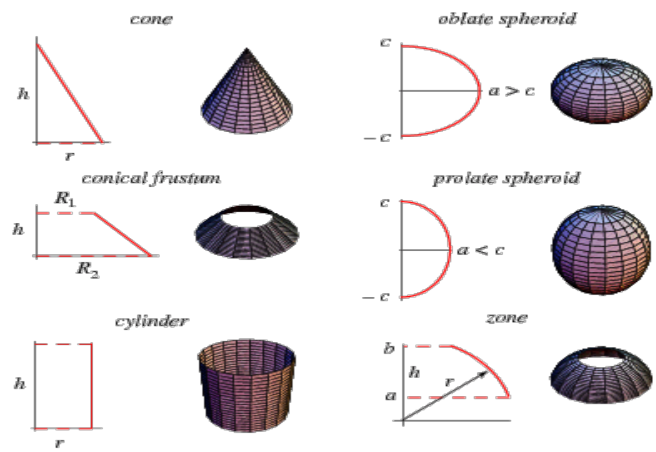
Analogically to parametric curves $(x,y,z)=(f(t),g(t),h(t))$, one defines parametric surfaces $(x,y,z)=(f(u,v),g(u,v),h(u,v))$ or in vector form $\vec{r}(u,v)=f(u,v)\mathrm{i}+g(u,v)\mathrm{j}+h(u,v)\mathrm{k}$.
More analogies, when we wanted the description of circle as function we used parametric representation, similarly a sphere can be described as parametric surface

$$\vec{r}(\varphi,\theta)=\langle a\sin\varphi\cos\theta,a\sin\varphi\sin\theta,a\cos\varphi\rangle$$

Unlike the circle, if we use $z=\pm\sqrt{x^2+y^2}$ to draw a sphere the result won't be as good image as with parametric surface representation, unless may be we use very fine grid.

In 2D a regular function y=f(x) were parameterized as (x,y)=(x,f(x)), similarly (for example) elliptic paraboloid $x=2y^2+3z^2+1$ can be expressed as $\vec{r}(y,z)=(2y^2+3z^2+1,y,z)$. Another example a cylinder $x^2+y^2=3$ can be expressed as $\vec{r}(\theta,z)=(\sqrt{3}\cos\theta,\sqrt{3}\sin\theta,z),0\le\theta,2\pi$.

Surface of Revolution, a curve y=f(x) rotated in the space can be described parametrically, for example $\langle x, f(x)\cos\theta, f(x)\sin\theta\rangle$



Thm: A plane can be parametrically described using two vectors $\vec{u}$ and $\vec{v}$ and a point P_0, all lying on the plane, in a following way. Any vector on this plane can always be described as a combination of two vectors as $\vec{x} = \overrightarrow{P_0P} = a\vec{u} + b\vec{v}$. With the use of point one define vector $\vec{r} = \overrightarrow{OP_0}$ and get the plane as $\vec{R} = \overrightarrow{OP_0} + \overrightarrow{P_0P} = r + a\vec{u} + b\vec{v}$

Proof: Rewrite

$$\vec{R} = \langle x,y,z\rangle = \langle r_1,r_2,r_3\rangle + a\langle u_1,u_2,u_3\rangle + b\langle v_1,v_2,v_3\rangle = \langle r_1+au_1+bv_1, r_2+au_2+bv_2, r_3+au_3+bv_3\rangle,$$

substitute it into regular plane equation, which we rewrite in a more convenient form to reach the usual plane equation:

$$0 = Ax+By+Cz-D = A(r_1+au_1+bv_1)+B(r_2+au_2+bv_2)+C(r_3+au_3+bv_3)-D =$$
$$= a(Au_1+Bu_2+Cu_3)+b(Av_1+Bv_2+Cv_3)+(Ar_1+Br_2+Cr_3)-(At_1+Bt_2+Ct_3) =$$
$$= a\langle A,B,C\rangle\cdot\vec{u} + b\langle A,B,C\rangle\cdot\vec{v} + \langle A,B,C\rangle\cdot(\vec{r}-\vec{t}) - D = \vec{n}\cdot\left((a\vec{u}+b\vec{v})-(\vec{t}-\vec{r})\right) = \vec{n}\cdot(\vec{x}-\vec{x}_0)$$

Ex 6. Find a vector representation of a plane determined by points $P_0(7,0,1), P_1(3,1,5), P_2(0,1,3)$ and then use it to find $\vec{n} = const\langle 2,20,-3\rangle$

We define vectors $\begin{matrix}\overrightarrow{P_0P_1} = \langle 3-7, 1-0, 5-1\rangle = \langle -4,1,4\rangle \\ \overrightarrow{P_0P_2} = \langle 0-7, 1-0, 3-2\rangle = \langle -7,1,2\rangle\end{matrix}$ and get

$$\vec{R} = \overrightarrow{OP_0} + s\overrightarrow{P_0P_2} + t\overrightarrow{P_0P_2} = \langle 7,0,1\rangle + s\langle -4,1,4\rangle + t\langle -7,1,2\rangle = \langle 7-4s-7t, s+t, 1+4s+2t\rangle$$

We rewrite it as

$$Ax+By+Cz-D=0 \Rightarrow A(7-4s-7t)+B(s+t)+C(1+4s+2t)-D =$$
$$=(7A+C)+(B+4C-4A)s+(B+2C-7A)t-D=0$$

to get 3 equation for 4 variables

$$\begin{cases}7A+C-D=0\\ B+4A-4C=0\\ B+7A-2C=0\end{cases} \Rightarrow \begin{cases}D=7A+C\\ B=4A-4C\\ B=7A-2C\end{cases} \Rightarrow \begin{cases}D=7A-\frac{3}{2}A=\frac{11}{2}A\\ B=4A+4\cdot\frac{3}{2}A=10A\\ 4A-4C=7A-2C \Rightarrow 3A=-2C \Rightarrow C=-\frac{3}{2}A\end{cases}$$

which mean one can be chosen arbitrary $\langle A,B,C\rangle = A\langle 1,10,-\frac{3}{2}\rangle_{A=2} = \langle 2,20,-3\rangle$.