3.3 Arc Length and Curvature (10.3)

3.3.1 Arc length

Def: An Arc Length of a curve described by vector function $\vec{v}(t)=\langle v_1(t),v_2(t),v_3(t)\rangle$ for $a\le t\le b$ is defined by $L=\int_a^b\left|\vec{v}\,'(t)\right|dt=\int_a^b\sqrt{v'_1(t)^2+v'_2(t)^2+v'_3(t)^2}\,dt$. **Note**: this is similar to the definition we learned in calculus I for parameterized curve in 2D.

Ex 1. Find arc length of $\vec{r}(t)=\langle\cos t-\sin t,\cos t+\sin t,2\sqrt{t}\rangle$

$$r'(t)=\langle-\sin t-\cos t,-\sin t+\cos t,\tfrac{1}{\sqrt{t}}\rangle$$

$$|r'(t)|=\sqrt{(\cos t+\sin t)^2+(\cos t-\sin t)^2+\tfrac{1}{t}}=\sqrt{2+\tfrac{1}{t}}$$

$$L=\int\sqrt{2+\tfrac{1}{t}}\,dt=\int\tfrac{\sqrt{2t+1}}{\sqrt{t}}\,dt\underset{\substack{x=\sqrt{t}\\ dx=\frac{1}{2\sqrt{t}}dt;\ \ 2x^2+1=2t+1}}{=}\int\sqrt{2x^2+1}\cdot 2\,dx=2\sqrt{2}\int\sqrt{\tfrac{1}{2}+x^2}\,dx=$$

$$\overset{\substack{\text{table of integrals}\\ \text{eq 21, }a^2=1/2}}{=}2\sqrt{2}\left(\tfrac{x}{2}\sqrt{\tfrac{1}{2}+x^2}+\tfrac{1}{4}\ln\left(x+\sqrt{\tfrac{1}{2}+x^2}\right)\right)+C=2\sqrt{2}\left(\tfrac{\sqrt{t}}{2}\sqrt{\tfrac{1}{2}+t}+\tfrac{1}{4}\ln\left(\sqrt{t}+\sqrt{\tfrac{1}{2}+t}\right)\right)+C$$

Arc length parameterization

The same curve can be described by different parameterizations, for example $\langle t,\sqrt{t},\sin t\rangle$ for $0\le t\le\pi$ describe the same curve as $\langle\pi t,\sqrt{t\pi},\sin\pi t\rangle$ for $0\le t\le 1$. A natural parameterization for a space curve is with respect to arc length. Thus, if a curve is parameterized with parameter t, i.e. as $\vec{r}(t)$, then we can find an arc length function as $s(t)=\int_0^t\left|\vec{r}\,'(t)\right|dt$ and try to re-parameterize it using $t=t(s)$. The distance along a space curve is independent of parameterization. This simply means that the total distance traveled along a curve is independent of the speed.

Note: It is usually difficult to find explicit formula for an arc length parameterization of a general curve. Fortunately, it can be used and have many applications even without explicit formula.

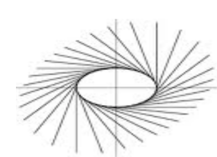
Ex 2. Find arc length parameterization of $\vec{r}(t)=\langle\cos t-\sin t,\cos t+\sin t,0\rangle$

$$\left|\vec{r}(t)=\sqrt{(\cos t-\sin t)^2+(\cos t+\sin t)^2}\right|=\sqrt{2}$$

$$s(t)=\int_0^t\sqrt{2}\,dt=\sqrt{2}t\Rightarrow t=\frac{s}{\sqrt{2}}\Rightarrow\vec{r}(t(s))=\left\langle\cos\frac{s}{\sqrt{2}}-\sin\frac{s}{\sqrt{2}},\cos\frac{s}{\sqrt{2}}+\sin\frac{s}{\sqrt{2}},0\right\rangle$$

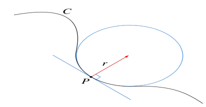
3.3.2 Curvature

Def: A parameterization $\vec{r}(t)$ is called smooth on an interval I if $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t)\neq 0$ on I. A curve is called smooth if it has a smooth parameterization.



Def: The **unit tangent vector** of a smooth curve $\vec{r}(t)$ is defined by $\vec{T}(t)=\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

Note: $\vec{T}(t)$ changes direction slowly when the curve is straight\flat and when the curve is sharp it change the direction faster.



Def: A curvature of a curve $\vec{r}(t)$ is measure how quickly the curve change direction at given point. A curvature can be understood as a reciprocal radius of inscribed\osculated circle tangent to the curve, also called circle of curvature. The curvature is often denoted as κ (kappa) and it given by $\kappa=\left|\frac{d\vec{T}}{ds}\right|$ where $\vec{T}(t)$ is a **unit tangent vector** and s is arc length parameter (so the curvature is independent of parameterization).

Since $\frac{d\vec{T}}{dt}=\frac{d\vec{T}}{ds}\frac{ds}{dt}$, one writes $\kappa=\left|\frac{d\vec{T}}{ds}\right|=\left|\frac{d\vec{T}/dt}{ds/dt}\right|$. The derivative of the formula of the arc length parameter $s(t)=\int_0^t|\vec{r}\,'(t)|dt$ gives $\frac{ds}{dt}=|\vec{r}\,'(t)|$, therefore we reformulate the curvature as $\kappa(t)=\frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$

Thm: $\kappa(t)=\frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

Proof: $\vec{T}(t)=\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\Rightarrow\vec{r}\,'(t)=|\vec{r}\,'(t)|\vec{T}(t)=\frac{ds}{dt}\vec{T}(t)=s'(t)\vec{T}(t)\Rightarrow\vec{r}\,''(t)=s''(t)\vec{T}(t)+s'(t)\vec{T}\,'(t)$

$$\begin{aligned}\vec{r}\,'(t)\times\vec{r}\,''(t)&=s'(t)\vec{T}(t)\times\left(s''(t)\vec{T}(t)+s'(t)\vec{T}\,'(t)\right)=\\&=s'(t)\vec{T}(t)\times s''(t)\vec{T}(t)+s'(t)\vec{T}(t)\times s'(t)\vec{T}\,'(t)=\left(s'(t)\right)^2\left(\vec{T}(t)\times\vec{T}\,'(t)\right)\end{aligned}$$

Note that $s'(t)\vec{T}(t)\times s''(t)\vec{T}(t)=0$ because $s'(t)\vec{T}(t)$ and $s''(t)\vec{T}(t)$ proportional vectors.
In order to get the following result

$$|\vec{r}\,'(t)\times\vec{r}\,''(t)|=\left(s'(t)\right)^2|\vec{T}(t)\times\vec{T}\,'(t)|=\left(s'(t)\right)^2|\vec{T}(t)||\vec{T}\,'(t)||\sin\alpha|=\left(s'(t)\right)^2|\vec{T}\,'(t)||\sin\alpha|,$$

where α is an angle between $\vec{T}(t)$ and $\vec{T}\,'(t)$.
We will later show that $\vec{T}(t)$ and $\vec{T}\,'(t)$ are orthogonal, i.e. $\alpha=\pi/2$ to get

$$|\vec{r}\,'(t)\times\vec{r}\,''(t)|=\left(s'(t)\right)^2|\vec{T}\,'(t)|\text{ or }|\vec{T}\,'(t)|=\frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{\left(s'(t)\right)^2}=\frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|^2}$$

$$\text{thus }\kappa=\frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}=\frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

Regarding orthogonally:

$$\vec{T}(t)\cdot\vec{T}(t)=|\vec{T}(t)|^2=1\Rightarrow 0=\frac{d}{dt}\left(\vec{T}(t)\cdot\vec{T}(t)\right)=\vec{T}\,'(t)\cdot\vec{T}(t)+\vec{T}(t)\cdot\vec{T}\,'(t)=2\vec{T}\,'(t)\cdot\vec{T}(t)$$

Therefore $\vec{T}\,'(t)\cdot\vec{T}(t)=0$ and $\alpha=\arccos\frac{\vec{T}\,'(t)\cdot\vec{T}(t)}{|\vec{T}\,'(t)||\vec{T}(t)|}=\arccos 0=\frac{\pi}{2}$

Ex 3. Find a curvature of the helix $\vec{r}(t)=\langle \cos t,\sin t,t\rangle$

$$\kappa=\frac{|\vec{r}(t)\times\vec{r}\,'(t)|}{|\vec{r}\,'(t)|^3}=\frac{|\langle -\sin t,\cos t,1\rangle\times\langle -\cos t,-\sin t,0\rangle|}{|\langle -\sin t,\cos t,1\rangle|^3}=\frac{|\langle \sin t,-\cos t,\sin^2 t+\cos^2 t\rangle|}{(\sin^2 t+\cos^2 t+1)^{3/2}}=$$

$$=\frac{\sqrt{\sin^2 t+\cos^2 t+1}}{2^{3/2}}=\frac{\sqrt{2}}{2\sqrt{2}}=\frac{1}{2}$$

Ex 4. Find an expression of curvature of the curve given by y=f(x)

We will rewrite the curve as $\vec{r}(t)=\langle x,f(x),0\rangle$, therefore

$$\vec{r}\,'(t)=\langle 1,f'(x),0\rangle$$

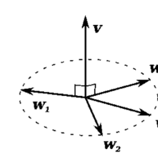
$$\vec{r}\,''(t)=\langle 0,f''(x),0\rangle$$

$$|\vec{r}\,'(t)\times\vec{r}\,''(t)|=|\langle 1,f'(x),0\rangle\times\langle 0,f''(x),0\rangle|=|\langle f'(x)\cdot 0-0\cdot f''(x),0\cdot 0-1\cdot 0,1\cdot f''(x)-f'(x)\cdot 0\rangle|=$$

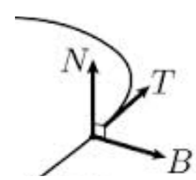
$$=|\langle 0,0,f''(x)\rangle|=\sqrt{(f''(x))^2}=|f''(x)|$$

Thus $\kappa=\dfrac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}=\dfrac{\|\langle 1,f'(x),0\rangle\times\langle 0,f''(x),0\rangle\|}{|\langle 1,f'(x)\rangle|^3}=\dfrac{|f''(x)|}{\left(1+(f'(x))^2\right)^{3/2}}$

3.3.3 The Normal and Binormal Vectors



There is more then one vector orthogonal to the tangent vector $\vec{T}(t)$. We already found that $\vec{T}\,'(t)$ is orthogonal to $\vec{T}(t)$. Note that $\vec{T}\,'(t)$ isn't unit vector. We define a (**principle**) **unit normal vector** to be $\vec{N}(t)=\dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$. We define additional vector, binormal vector by $\vec{B}(t)=\vec{T}(t)\times\vec{N}(t)$ which is orthogonal to both $\vec{T}(t)$ and $\vec{N}(t)$, and is also a unit vector, since $|\vec{B}(t)|=|\vec{T}(t)||\vec{N}(t)|=1$.



A plane determined by $\vec{N}(t)$ and $\vec{B}(t)$ is called normal plane, a plane determined by $\vec{T}(t)$ and $\vec{N}(t)$ is called osculating plane and is related to the circle of curvature.

Ex 5. Find unit normal & binormal vector for $\vec{r}(t)=\langle \sin 2t,t,\cos 2t\rangle$

$$\vec{T}(t)=\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}=\frac{\langle 2\cos 2t,1,-2\sin 2t\rangle}{\sqrt{\sin^2 2t+1+\cos^2 2t}}=\frac{1}{\sqrt{2}}\langle 2\cos 2t,1,-2\sin 2t\rangle$$

$$\vec{T}\,'(t)=-\frac{4}{\sqrt{2}}\langle \sin 2t,0,\cos 2t\rangle=-2\sqrt{2}\langle \sin 2t,0,\cos 2t\rangle$$

$$\vec{N}(t)=\frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}=\frac{-2\sqrt{2}\langle \sin 2t,0,\cos 2t\rangle}{2\sqrt{2}}=-\langle \sin 2t,0,\cos 2t\rangle$$

$$\vec{B}(t)=\vec{T}(t)\times\vec{N}(t)=\tfrac{1}{\sqrt{5}}\langle 2\cos 2t,1,-2\sin 2t\rangle\times\left(-\langle \sin 2t,0,\cos 2t\rangle\right)=$$

$$=-\tfrac{1}{\sqrt{5}}\left\langle \cos 2t+0,-2\sin^2 2t-2\cos^2 2t,0-\sin 2t\right\rangle=\tfrac{1}{\sqrt{5}}\langle -\cos 2t,2,\sin 2t\rangle$$