3.2 Derivatives and Integrals of Vector Functions (10.2)

Def: Derivative of a vector function is defined by $\frac{d\vec{f}}{dt}=\vec{f}'(t)=\lim_{h\to 0}\frac{\vec{f}(t+h)-\vec{f}(t)}{h}$

If we define: $\vec{f}(t)=\langle f_1(t),f_2(t),f_3(t)\rangle$ we get

Thm:

$$\vec{f}'(t)=\lim_{h\to 0}\frac{\vec{f}(t+h)-\vec{f}(t)}{h}=\lim_{h\to 0}\frac{\langle f_1(t+h),f_2(t+h),f_3(t+h)\rangle-\langle f_1(t),f_2(t),f_3(t)\rangle}{h}=$$

$$=\lim_{h\to 0}\left\langle\frac{f_1(t+h)-f_1(t)}{h},\frac{f_2(t+h)-f_2(t)}{h},\frac{f_3(t+h)-f_3(t)}{h}\right\rangle=\langle f_1'(t),f_2'(t),f_3'(t)\rangle$$

Ex 8. $\frac{d}{dt}\langle t,t^2,t^3\rangle=\langle 1,2t,3t^2\rangle$

Ex 9. $\vec{r}(t)=\cos t\,\mathrm{i}+\sin t\,\mathrm{j}+t\,\mathrm{k}\Rightarrow\vec{r}'(t)=-\sin t\,\mathrm{i}+\cos t\,\mathrm{j}+\mathrm{k}$

Def: A vector tangent to a curve $\vec{r}(t)$ at point t_0 is given by $\vec{r}'(t_0)$ provided that $\vec{r}(t)$ exists at t_0 and that $\vec{r}'(t_0)\neq 0$. We often interested in unit tangent vector defined by $T(t_0)=\frac{\vec{r}'(t_0)}{|\vec{r}'(t_0)|}$.

Ex 10. Find unit tangent vector to $\vec{r}(t)=\cos t\,\mathrm{i}+\sin t\,\mathrm{j}+t\,\mathrm{k}$ at $t_0=0$:

$$\vec{r}'(t)=-\sin t\,\mathrm{i}+\cos t\,\mathrm{j}+\mathrm{k}\Rightarrow$$

$$T(t_0)=\frac{\vec{r}'(t_0)}{|\vec{r}'(t_0)|}=\frac{\langle -\sin t,\cos t,1\rangle}{|\sin^2 t+\cos^2 t+1|}=\frac{1}{2}\langle -\sin t,\cos t,1\rangle$$

Differentiating rules:

1. $(\vec{u}(t)+\vec{v}(t))'=\vec{u}'(t)+\vec{v}'(t)$
2. $(c\vec{u}(t))'=c\vec{u}'(t)$
3. $$\begin{aligned}(f(t)\vec{v}(t))' &=\frac{d}{dt}\langle f(t)v_1(t),f(t)v_2(t),f(t)v_3(t)\rangle=\\ &=\langle f'(t)v_1(t)+f(t)v'_1(t),f'(t)v_2(t)+f(t)v'_2(t),f'(t)v_3(t)+f(t)v'_3(t)\rangle\\ &=f'(t)\vec{v}(t)+f(t)\vec{v}'(t)\end{aligned}$$
4. $$\begin{aligned}(\vec{u}(t)\cdot\vec{v}(t))' &=\frac{d}{dt}(u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t))=\\ &=u'_1(t)v_1(t)+u_1(t)v'_1(t)+u'_2(t)v_2(t)+u_2(t)v'_2(t)+u'_3(t)v_3(t)+u_3(t)v'_3(t)=\\ &=\vec{u}'(t)\cdot\vec{v}(t)+\vec{u}(t)\cdot\vec{v}'(t)\end{aligned}$$

5. $$\begin{aligned}(\vec{u}(t)\times\vec{v}(t))' &= \frac{d}{dt}\left\langle u_2(t)v_3(t)-u_3(t)v_2(t),\bullet,\bullet\right\rangle = \\ &= \left\langle u'_2(t)v_3(t)+u_2(t)v'_3(t)-\left(u'_3(t)v_2(t)+u_3(t)v'_2(t)\right),\bullet,\bullet\right\rangle \\ &= \vec{u}'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}'(t)\end{aligned}$$

6. Chain rule:

$$\begin{aligned}(\vec{u}(f(t)))' &= \frac{d}{dt}\left\langle u_1(f(t)),u_2(f(t)),u_3(f(t))\right\rangle = \left\langle u'_1(f(t))f'(t),u'_2(f(t))f'(t),u'_3(f(t))f'(t)\right\rangle = \\ &= \vec{u}'(f(t))f'(t)\end{aligned}$$

Ex 11. $$\begin{aligned}&\frac{d}{dt}\left(\left\langle t,e^t,0\right\rangle\times\left\langle 0,t,\cos t\right\rangle\right) = \frac{d}{dt}\left\langle e^t\cos t,-t\cos t,t^2\right\rangle = \left\langle e^t(\cos t-\sin t),-\cos t+t\sin t,2t\right\rangle \\ &\frac{d}{dt}\left(\left\langle t,e^t,0\right\rangle\times\left\langle 0,t,\cos t\right\rangle\right) = \left(\frac{d}{dt}\left\langle t,e^t,0\right\rangle\right)\times\left\langle 0,t,\cos t\right\rangle+\left\langle t,e^t,0\right\rangle\times\left(\frac{d}{dt}\left\langle 0,t,\cos t\right\rangle\right) = \\ &= \left\langle 1,e^t,0\right\rangle\times\left\langle 0,t,\cos t\right\rangle+\left\langle t,e^t,0\right\rangle\times\left\langle 0,1,-\sin t\right\rangle = \left\langle e^t(\cos t-\sin t),-\cos t+t\sin t,2t\right\rangle\end{aligned}$$

Ex 12. $$\begin{aligned}&\frac{d}{dt}\left\langle t^2,\cos t^2,\ln t^2\right\rangle = 2t\left(\frac{d}{dp}\left\langle p,\cos p,\ln p\right\rangle\right)_{p=t^2} = 2t\left\langle 1,-\sin p,\frac{1}{p}\right\rangle_{p=t^2} = \\ &= 2t\left\langle 1,-\sin t^2,\frac{1}{t^2}\right\rangle = \left\langle 2t,-2t\sin t^2,\frac{2}{t}\right\rangle\end{aligned}$$

3.2.1 Integrals

$$\begin{aligned}&\int_a^b \vec{r}(t)dt = \lim_{n\to\infty}\sum_{j=1}^{n}\vec{r}\left(t_j^*\right)\Delta t = \lim_{n\to\infty}\sum_{j=1}^{n}\left(r_1\left(t_j^*\right)\vec{i}+r_2\left(t_j^*\right)\vec{j}+r_3\left(t_j^*\right)\vec{k}\right)\Delta t = \\ &= \lim_{n\to\infty}\left(\sum_{j=1}^{n}r_1\left(t_j^*\right)\Delta t\right)\vec{i}+\lim_{n\to\infty}\left(\sum_{j=1}^{n}r_2\left(t_j^*\right)\Delta t\right)\vec{j}+\lim_{n\to\infty}\left(\sum_{j=1}^{n}r_3\left(t_j^*\right)\Delta t\right)\vec{k} = \\ &= \left(\int_a^b r_1(t)dt\right)\vec{i}+\lim_{n\to\infty}\left(\int_a^b r_2(t)dt\right)\vec{j}+\lim_{n\to\infty}\left(\int_a^b r_3(t)dt\right)\vec{k} = \left\langle\int_a^b r_1(t)dt,\int_a^b r_2(t)dt,\int_a^b r_3(t)dt\right\rangle\end{aligned}$$

Ex 13. $$\int\left\langle t^2,\cos t,\frac{1}{t}\right\rangle dt = \left\langle\int t^2\,dt,\int\cos t\,dt,\int\frac{dt}{t}\right\rangle = \left\langle\frac{t^3}{3},\sin t,\ln t\right\rangle$$