3 Vector Functions

3.1 Vector Functions and Space Curves (10.1)

We have learned by now functions of one variable (and a little about functions of two variables). All these function were real valued function $f:R\to R$ (and $f:R^2\to R$), thus their result was a real number. Since we recently learned a (new) concept of vectors, we can now generalize functions to a vector valued functions. Such function are not totally new thing, we already met one of this kind (which one?).

Def: A vector-valued function is a function whose domain is a set of real numbers and a range is a set of vectors. We denote it as $f:R\to R^2$ if a range of the function is set of 2D vectors or $f:R\to R^3$ if the range is set of 3D vectors.

Note: we are most interested in $f:R\to R^3$

Examples:

1. General case: $r(t)=\langle f(t),g(t),h(t)\rangle$
2. A linear function or a line: $\vec{l}(t)=\langle a_1t+b_1,a_2t+b_2,a_3t+b_3\rangle$
3. Parabola in xy plane $r(t)=\langle t,t^2,0\rangle$
4. A twisted cubic: $r(t)=t\,\mathrm{i}+t^2\,\mathrm{j}+t^3\,\mathrm{k}$
5. A helix $r(t)=\cos t\,\mathrm{i}+\sin t\,\mathrm{j}+t\,\mathrm{k}$
6. An opposite direction, different start point helix $r(t)=\sin t\,\mathrm{i}+\cos t\,\mathrm{j}+t\,\mathrm{k}$
7. Spiral along y-axis $r(t)=\sin\mathrm{i}+t\,\mathrm{j}+\cos t\,\mathrm{k}$
8. Tornado $\langle t\cos t,t\sin t,t\rangle$

Def: A limit of vector-valued function $r(t)=\langle f(t),g(t),h(t)\rangle$ is given by $\lim\limits_{t\to a}r(t)=\left\langle \lim\limits_{t\to a}f(t),\lim\limits_{t\to a}g(t),\lim\limits_{t\to a}h(t)\right\rangle$ provided the limits of the component function exist.

Ex 1. $$\lim_{t\to\pi/2}\langle t,\sin t,5t+1\rangle=\left\langle\frac{\pi}{2},1,\frac{5\pi}{2}+1\right\rangle$$

Ex 2. $$\lim_{t\to0}\left\langle t^2,\frac{\sin t}{t},\ln t\right\rangle=\left\langle\lim_{t\to0}t^2,\lim_{t\to0}\frac{\sin t}{t},\lim_{t\to0}\ln t\right\rangle=\langle0,1,-\infty\rangle\text{ - divergent}$$

Ex 3.
$$\lim_{t\to0}\left\langle(1+2t)^{1/t},t\ln t,(1+1/t)^t\right\rangle=\left\langle\lim_{t\to0}(1+2t)^{1/t},\lim_{t\to0}\frac{\ln t}{1/t},\lim_{t\to0}(1+1/t)^t\right\rangle=$$
$$=\left\langle e^{\lim\limits_{t\to0}\ln(1+2t)^{1/t}},\lim_{t\to0}\frac{1/t}{-1/t^2},e^{\lim\limits_{t\to0}\ln(1+1/t)^t}\right\rangle=\left\langle e^{\lim\limits_{t\to0}\frac{\ln(1+2t)}{t}},-\lim_{t\to0}t,e^{\lim\limits_{t\to0}t\ln(1+1/t)}\right\rangle$$
$$=\left\langle e^{\lim\limits_{t\to0}\frac{2/(1+2t)}{1}},0,e^{\lim\limits_{t\to0}\frac{\ln(1+1/t)}{1/t}}\right\rangle=\left\langle e^2,0,e^{\lim\limits_{t\to0}\frac{(-1/t^2)/(1+1/t)}{-1/t^2}}\right\rangle=\left\langle e^2,0,e^{\lim\limits_{t\to0}\frac{t}{t+1}}\right\rangle=\langle e^2,0,1\rangle$$

Ex 4. Determine the shape of the intersection between cylinder $x^2+(y-1)^2=2$ and a plane $y+z=4$

We know that projection of $x^2+(y-1)^2=2$ on xy plane is a circle of radius $\sqrt{2}$ centered at $(0,1)$ therefore we can rewrite it as $x=\sqrt{2}\cos t, y-1=\sqrt{2}\sin t, 0\le t\le 2\pi$. We plug it in the equation of the plane solved for z to get $y+z=4 \Rightarrow z=4-y=4-\left(\sqrt{2}\sin t+1\right)=3-\sqrt{2}\sin t$ for $0\le t\le 2\pi$.
Thus $\vec{r}(t)=\left\langle \sqrt{2}\cos t, \sqrt{2}\sin t+1, 3-\sqrt{2}\sin t\right\rangle$ for $0\le t\le 2\pi$.
This curve is an ellipse since

$$\frac{x^2}{2}+\frac{(y-1)^2}{4}+\frac{(z-3)^2}{2}=\cos^2 t+\frac{\sin^2 t}{2}+\frac{\sin^2 t}{2}=1$$

Ex 5. Find expression of a unit circle on a $y+z=4$ plane.

We need to find $v=\langle v_1,v_2,v_3\rangle$ such that $v_1^2+v_2^2+v_3^2=1$ and $v_2+v_3=4$, i.e. $v=\langle v_1,v_2,4-v_2\rangle$ and $v_1^2+v_2^2+(4-v_2)^2=1$, thus $x^2+y^2+(4-y)^2=1$.

Ex 6. Find formula of the line that connect between points $P_0(1,2)$ and $P_1(3,5)$

An "old" approach: 1) find a slope $m=\frac{y_1-y_0}{x_1-x_0}=\frac{5-2}{3-1}=\frac{3}{2}$ 2) use line formula

$$(y-y_0)=(x-x_0)m \Rightarrow y-2=(x-1)\frac{3}{2} \Rightarrow y=\frac{3}{2}x-\frac{3}{2}+2=\frac{3}{2}x+\frac{1}{2}$$

A vector approach from 9.5:

$$l(t)=(1-t)\langle 1,2\rangle+t\langle 3,5\rangle=\langle 1-t+3t, 2-2t+5t\rangle=\langle 1+2t, 2+3t\rangle$$

One can verify this equation is similar to previous one

$$x=1+2t \Rightarrow t=\frac{x-1}{2}$$

$$y=2+3t=2+3\cdot\frac{x-1}{2}=2+\frac{3}{2}x-\frac{3}{2}=\frac{3}{2}x+\frac{1}{2}$$

Ex 7. Find a vector line equation between $P_0(1,0,-1)$ and $P_1(2,1,7)$

$$l(t)=(1-t)\langle 1,0,-1\rangle+t\langle 2,1,7\rangle=\langle 1-t+2t, 0+t, t-1+7t\rangle=\langle 1+t, t, 8t-1\rangle$$