2.5 Equations of Lines and Planes (9.5)

Recall: parametric curves $(x,y)=(f(t),g(t)),\quad a\le t\le b$ describe the curve by describing each point in the xy plane

Ex 1. Circle: $(x,y)=(\cos t,\sin t),\quad 0\le t\le 2\pi$

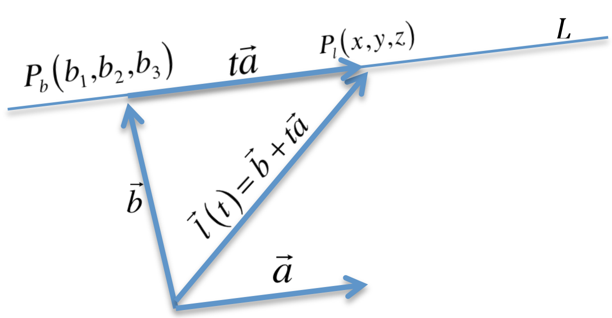
The formula of line in 2D is often given by equation $y=ax+b$ in a parametric form one can express it as $(x,y)=(t,at+b),\quad -\infty\le t\le\infty$.

One can reformulate line as $\tilde{A}y+\tilde{B}=Ax+B$, which can put it in a previous form using $a=\frac{A}{\tilde{A}},b=\frac{B-\tilde{B}}{\tilde{A}}$, i.e. $y=ax+b=\frac{A}{\tilde{A}}x+\frac{B-\tilde{B}}{\tilde{A}}$. In the parametric form, one writes $(x,y)=(\tilde{A}t+\tilde{B},At+B)$.

Similarly one expand parametric curves to 3D as $(x,y,z)=(f(t),g(t),h(t)),\quad a\le t\le b$ and the line is given as $(x,y,z)=(a_1t+b_1,a_2t+b_2,a_3t+b_3)$.

One expresses parametric curves in vector form using $\vec{r}(t)=\langle f(t),g(t),h(t)\rangle$ and consequently the line is given by $\vec{l}(t)=\langle a_1t+b_1,a_2t+b_2,a_3t+b_3\rangle=t\langle a_1,a_2,a_3\rangle+\langle b_1,b_2,b_3\rangle=\vec{a}t+\vec{b}$.

It is possible to explain this expression of the line as following. Let $P_b(b_1,b_2,b_3)$ be a point on a line L and $\vec{a}$ be a vector parallel to this line. Let O denote the origin and choose another point on the line L: $P_l(x,y,z)$. Next define vectors $\overrightarrow{OP_b}=\vec{b}=\langle b_1,b_2,b_3\rangle$ and $\overrightarrow{OP_l}=\vec{l}=\langle x_1,x_2,x_3\rangle$. One writes $\vec{l}=\vec{b}+\overrightarrow{P_bP_l}$. But $\overrightarrow{P_bP_l}$ is vector parallel to the line L and therefore is parallel to vector $\vec{a}$, which mean $\overrightarrow{P_bP_x}=t\vec{a}$ where t is a scalar. Finally we get $\vec{l}(t)=\vec{b}+t\vec{a}$.



Yet another way to express line in 3D is by

$$(x,y,z)=(x_0+at,y_0+bt,z_0+ct)\Rightarrow\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}(=t)$$

Ex 2. Find line trough point (1,2,3) parallel to vector $\langle 1,0,-1\rangle$:

$$l(t)=\langle 1,2,3\rangle+t\langle 1,0,-1\rangle=\langle 1+t,2,3-t\rangle$$

Ex 3. Find the line between points, (1,2,3) and (3,2,1): we first define a vector $v=\langle 3-1,2-2,1-3\rangle=\langle 2,0,-2\rangle=2\langle 1,0,-1\rangle$ and we actually got the same line as before $l(t)=\langle 1,2,3\rangle+t\langle 1,0,-1\rangle=\langle 1+t,2,3-t\rangle$

Ex 4. Show that the lines $l_1(t)=\langle -1,2,2\rangle t+\langle 1,0,2\rangle$ and $l_2(t)=\langle 1,1,-1\rangle t+\langle 0,1,1\rangle$ are **Skew lines**, that is not parallel and not intersecting:

The parallel lines should have similar, a.k.a. proportional, direction (vectors **a** in **a**t+**b** form). That is $\langle -1,2,2\rangle=const\langle 1,1,-1\rangle$, but $\frac{-1}{1}\neq\frac{2}{1}\neq\frac{2}{-1}$, so they aren't parallel.

Intersecting lines should have a solution to an equation $l_1(t)=l_2(t)$, thus there should be s,t that satisfy $\begin{cases}-t+1=s\\ 2t=s+1\\ 2t+2=-s+1\end{cases}$, but if we solve the first 2 we get

$$\begin{cases}-t+1=s\\ 2t=s+1\end{cases}\Rightarrow\begin{cases}1-t=s\\ 2t-1=s\end{cases}\Rightarrow 1-t=2t-1\Rightarrow 3t=2\Rightarrow\begin{cases}t=2/3\\ s=1/3\end{cases}$$

which doesn't satisfy the 3rd one, that is $4/3+2\neq -1/3+1=2/3$.
Thus, the lines l_1,l_2 aren't intersects and aren't parallel, that is they are skew.

2.5.1 Planes

At the beginning of this chapter we discussed the formula of the plane:

$$Ax+By+Cz=D$$

one rewrite it in the following form $\langle A,B,C\rangle\cdot\langle x,y,z\rangle=D$ or in the more classical way

$$\langle A,B,C\rangle\cdot(\langle x,y,z\rangle-\langle x_0,y_0,z_0\rangle)=0 \text{ or } \langle A,B,C\rangle\cdot\langle x,y,z\rangle=\langle A,B,C\rangle\cdot\langle x_0,y_0,z_0\rangle$$

which still the same thing in general

$$\langle A,B,C\rangle\cdot(\langle x,y,z\rangle-\langle x_0,y_0,z_0\rangle)=Ax+By+Cz-(Ax_0+By_0+Cz_0)=Ax+By+Cz-D=0,$$

but note one important thing $\langle A,B,C\rangle$ is orthogonal to $(\langle x,y,z\rangle-\langle x_0,y_0,z_0\rangle)$.

We denote a normal $\vec{n}=\langle A,B,C\rangle$, points $P_0(x_0,y_0,z_0),P(x,y,z)$ and vector s $\vec{x}_0=\overrightarrow{OP_0}=\langle x_0,y_0,z_0\rangle$ and $\vec{x}=\overline{OP}=\langle x,y,z\rangle$

Def: The vector formula of the plane is given by $\vec{n}(\vec{x}-\vec{x}_0)=0$ or equivalently $\vec{n}\vec{x}=\vec{n}\vec{x}_0$.

Thus, in order to define plane we need a point on this plane and a normal to this plane. Compare to the line definition where we were need a point a vector parallel to that line.

Ex 5. Find a plane determined by a normal $\vec{n} = \langle 3,2,1 \rangle$ and a point $P_0 = (0,1,0)$

$$\vec{n}(\vec{x} - \vec{x}_0) = \langle 3,2,1 \rangle \cdot \langle x, y-1, z \rangle = 3x + 2y - 2 + z = 0 \text{ or } 3x + 2y + z = 2$$

Ex 6. Find a plane determined by points $P_0(7,0,1), P_1(3,1,5), P_2(0,1,3)$

We define vectors

$$\overrightarrow{P_0P_1} = \langle 3-7, 1-0, 5-1 \rangle = \langle -4,1,4 \rangle$$
$$\overrightarrow{P_0P_2} = \langle 0-7, 1-0, 3-2 \rangle = \langle -7,1,2 \rangle$$

and find a normal

$$\vec{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \langle -4,1,4 \rangle \times \langle -7,1,2 \rangle = \langle 1\cdot 2 - 4\cdot 1, 4\cdot(-7) - (-4)\cdot 2, -4\cdot 1 - 1(-7) \rangle = \langle -2,-20,3 \rangle$$

Finally the plane is given by

$$\langle -2,-20,3 \rangle \cdot \langle x-7, y, z-1 \rangle = -2x + 14 - 20y + 3z - 3 = -2x - 20y + 3z + 11 = 0 \text{ or } 2x + 20y - 3z = 11$$

Ex 7. Find intersection between the line $\vec{l}(t) = \langle 2,1,1 \rangle t + \langle 1,0,3 \rangle$ and the plane given by $\langle 1,2,-3 \rangle \cdot \langle x+2, y+1, z-1 \rangle = 0$.

We rewrite the line by parametric equation $(x,y,z) = (2t+1, t, t+3)$ and put it into equation of the plane

$$0 = \langle 1,2,-3 \rangle \cdot \langle 2t+1+2, t+1, t+3-1 \rangle = \langle 1,2,-3 \rangle \cdot \langle 2t+3, t+1, t+2 \rangle = 2t + 3 + 2t + 2 - 3t - 6 = t - 1$$

thus the intersection occurs at t=1 and therefore $(x,y,z) = (2t+1, t, t+3)_{t=1} = (1,1,3)$

Thm: The angle between planes is the angle between their normal vectors.

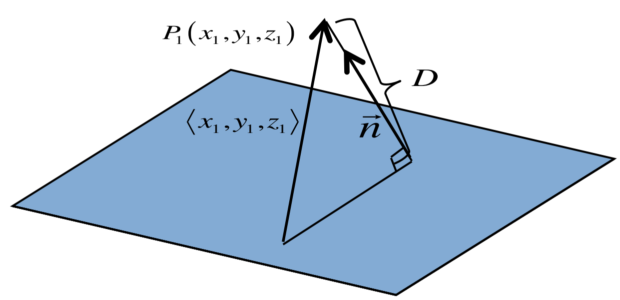
Ex 8. Find an angle between $x + y + z = 1$ and $x - y + 2z = 0$

The normal vectors of these planes are: $\vec{n}_1 = \langle 1,1,1 \rangle$ and $\vec{n}_2 = \langle 1,-1,2 \rangle$ and the angle between them is $\alpha = \arccos \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \arccos \frac{\langle 1,1,1 \rangle \cdot \langle 1,-1,2 \rangle}{\sqrt{1+1+1}\sqrt{1+1+4}} = \arccos \frac{2}{\sqrt{18}} \approx 1.07991$

Def: The distance between a point $P_1(x_1, y_1, z_1)$ and a plane $n_1x + n_2y + n_3z + d = 0$ is Is given by

$$D = \frac{|n_1x_1 + n_2y_1 + n_3z_1 + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

The vector $\langle x_1, y_1, z_1 \rangle$ may have any starting point on a plane.



Ex 9. Find distance from the point $P(1,2,3)$ to the plane $x - y + 2z = 0$

$$D = \frac{|1\cdot 1 + (-1)\cdot 2 + 2\cdot 3 + 0|}{\sqrt{1+1+4}} = \frac{5}{\sqrt{6}}$$

Notes:

- To find distance between parallel planes choose a point on one and use previous formula.
- To find distance between skew lines find the distance between their planes.