2.4 The Cross Product (9.4)

Def: A 2x2 matrix is an entity of the following form $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Similarly a 3x3 matrix has a form $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

Def: Every matrix is associated with a special number called **determinant**. The determinant of 2x2 matrix is given by $\det A=|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$. The determinant of 3x3 matrix is given in terms of determinants of 2x2 matrices as following

$$\det A=|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}=$$

$$=a_{11}\left(a_{22}a_{33}-a_{23}a_{32}\right)+a_{12}\left(a_{11}a_{33}-a_{13}a_{31}\right)+a_{13}\left(a_{21}a_{32}-a_{22}a_{31}\right)$$

Ex 7. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}=4-6=-2$

Ex 8. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}=1\cdot\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}-2\cdot\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}+3\cdot\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}=$

$=(5\cdot 9-6\cdot 8)-2(4\cdot 9-6\cdot 7)+3(4\cdot 8-5\cdot 7)=$

$=(45-48)-2(36-42)+3(32-35)=-3-2\cdot(-6)+3\cdot(-3)=-3+12-9=0$

Def: The cross product between vectors $\vec{a}=\langle a_1,a_2,a_3\rangle$ and $\vec{b}=\langle b_1,b_2,b_3\rangle$ is given by

$$\vec{a}\times\vec{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=\left(a_2b_3-a_3b_2\right)\mathbf{i}-\left(a_1b_3-a_3b_1\right)\mathbf{j}+\left(a_1b_2-a_2b_1\right)\mathbf{k}=$$

$$=\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\rangle$$

Ex 9. $\langle 1,2,3\rangle\times\langle 4,5,6\rangle=\langle 2\cdot 6-3\cdot 5,3\cdot 4-1\cdot 6,1\cdot 5-2\cdot 4\rangle=\langle 12-15,12-6,5-8\rangle=\langle -3,6,-3\rangle$

Ex 10. $\mathbf{i}\times\mathbf{j}=\langle 1,0,0\rangle\times\langle 0,1,0\rangle=\langle 0\cdot 0-0\cdot 1,0\cdot 0-1\cdot 0,1\cdot 1-0\cdot 0\rangle=\langle 0,0,1\rangle=\mathbf{k}$

Ex 11. $\mathbf{j}\times\mathbf{i}=\langle 0,1,0\rangle\times\langle 1,0,0\rangle=\langle 1\cdot 0-0\cdot 0,0\cdot 1-0\cdot 0,0\cdot 0-1\cdot 1\rangle=-\langle 0,0,1\rangle=-\mathbf{k}$

Similarly $\mathbf{j}\times\mathbf{k}=\mathbf{i}$ $\quad\mathbf{k}\times\mathbf{j}=-\mathbf{i}$ $\quad\mathbf{k}\times\mathbf{i}=\mathbf{j}$ $\quad\mathbf{i}\times\mathbf{k}=-\mathbf{j}$

Note that in general cross product is not commutative, i.e. $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$

Ex 12. $\mathbf{i}\times\mathbf{i}=\langle 1,0,0\rangle\times\langle 1,0,0\rangle=\langle 0\cdot 0-0\cdot 0,0\cdot 1-1\cdot 0,1\cdot 0-0\cdot 1\rangle=\vec{0}$

In general $\vec{a}\times\vec{a}=\langle a,b,c\rangle\times\langle a,b,c\rangle=\langle a_2a_3-a_3a_2,a_3a_1-a_1a_3,a_1a_2-a_2a_1\rangle=\vec{0}$ also if $b=k\vec{a}$ then $\vec{a}\times\vec{b}=0$

Ex 13. $(\mathbf{i}\times\mathbf{i})\times\mathbf{j}=0\times\mathbf{j}=0\neq\mathbf{i}\times(\mathbf{i}\times\mathbf{j})=\mathbf{i}\times\mathbf{k}=-\mathbf{j}$

In general $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\neq\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

Properties of Cross Product

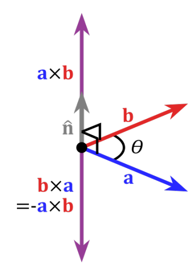
1) $\mathbf{a}\times\mathbf{b}=\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\rangle=-\langle a_3b_2-a_2b_3,a_1b_3-a_3b_1,a_2b_1-a_1b_2\rangle=-\mathbf{b}\times\mathbf{a}$

2) $(k\mathbf{a})\times\mathbf{b}=\langle(ka_2)b_3-(ka_3)b_2,(ka_3)b_1-(ka_1)b_3,(ka_1)b_2-(ka_2)b_1\rangle=$
$=k\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\rangle=k\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\rangle=k(\mathbf{a}\times\mathbf{b})=$
$=\mathbf{a}\times(k\mathbf{b})$

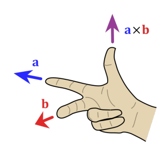
3) $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\langle a_2(b_3+c_3)-a_3(b_2+c_2),a_3(b_1+c_1)-a_1(b_3+c_3),a_1(b_2+c_2)-a_2(b_1+c_1)\rangle=$
$=\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\rangle+\langle a_2c_3-a_3c_2,a_3c_1-a_1c_3,a_1c_2-a_2c_1\rangle=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$

4) $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\langle(a_2+b_2)c_3-(a_3+b_3)c_2,(a_3+b_3)c_1-(a_1+b_1)c_3,(a_1+b_1)c_2-(a_2+b_2)c_1\rangle=$
$=\langle a_2c_3-a_3c_2,a_3c_1-a_1c_3,a_1c_2-a_2c_1\rangle+\langle b_2c_3-b_3c_2,b_3c_1-b_1c_3,b_1c_2-b_2c_1\rangle=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$

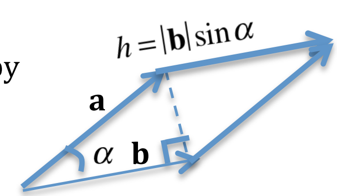
2.4.1 Geometric meaning of cross product



Def: Geometrical definition of cross product is given by $\vec{a}\times\vec{b}=\left(|\vec{a}||\vec{b}|\sin\alpha\right)\mathbf{n}$ where α is the angle between the vectors $\vec{a}$ and $\vec{b}$ and $\boldsymbol{n}$ is a unit vector perpendicular\orthogonal to both vectors $\vec{a}$ and $\vec{b}$.



Corollary: $\vec{a}\times\vec{b}$ orthogonal to both vectors $\vec{a}$ and $\vec{b}$.
Corollary: Non zero vectors $\vec{a}$ and $\vec{b}$ are parallel iff $\vec{a}\times\vec{b}=0$.



Thm: $|\vec{a}\times\vec{b}|=\left||\vec{a}||\vec{b}|\sin\alpha\right|$ is area of parallelogram determined by vectors $\vec{a}$ and $\vec{b}$.

Def: An angle between 2 vectors defined $|\sin\alpha|=\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$. Thus

$$0\le\alpha\le\pi:\alpha=\arcsin\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\qquad\qquad \pi\le\alpha\le 2\pi:\alpha=-\arcsin\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

2.4.2 Triple products

Def: **Scalar Triple product** of vectors $\vec{a},\vec{b},\vec{c}$ is defined to be $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$.

If we think about parallelepiped defined by vectors $\vec{a},\vec{b},\vec{c}$ with with a base of parallelogram defined by $\vec{b},\vec{c}$. The height of the parallelogram is $h=\left|\vec{a}\cos\theta\right|$ and the area of the base is $A=\left|\vec{b}\times\vec{c}\right|$.

Def: The volume parallelepiped defined by vectors $\vec{a},\vec{b},\vec{c}$ is given by:

$$V=hA=\left|\vec{a}\right|\cos\theta\left|\vec{b}\times\vec{c}\right|=\vec{a}\cdot\left(\vec{b}\times\vec{c}\right).$$

If we consider another side of the parallelepiped as base, say $\vec{a},\vec{b}$, then we get the following identity: $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$ (the last equality is due to commutative property of dot product).

Def: The **vector triple product is** given by $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$ (it will be useful in chapter 10)
The proof is long algebraicew work, you have it as a homework exercise, the hints for it are:
1) use component notations;
2) develop both sides using definitions to get them to the similar expression.

Ex 14. Let $\vec{a}=\langle 1,0,2\rangle,\vec{b}=\langle 0,1,1\rangle,\vec{c}=\langle 3,4,0\rangle$ compute

a. $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\left(\langle 1,0,2\rangle\times\langle 0,1,1\rangle\right)\cdot\langle 3,4,0\rangle=\langle 0\cdot 1-2\cdot 1,2\cdot 0-1\cdot 1,1\cdot 1-0\cdot 0\rangle\cdot\langle 3,4,0\rangle=$
$=\langle -2,-1,1\rangle\cdot\langle 3,4,0\rangle=-2\cdot 3+\left(-1\right)\cdot 4+1\cdot 0=-6-4=-10$

b. $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\langle 1,0,2\rangle\cdot\left(\langle 0,1,1\rangle\times\langle 3,4,0\rangle\right)=\langle 1,0,2\rangle\cdot\langle 1\cdot 0-1\cdot 4,1\cdot 3-0\cdot 0,0\cdot 4-1\cdot 3\rangle=$
$=\langle 1,0,2\rangle\cdot\langle -4,3,-3\rangle=1\cdot\left(-4\right)+0\cdot 3+2\cdot\left(-3\right)=-4+0-6=-10$

c. $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\langle 1,0,2\rangle\times\left(\langle 0,1,1\rangle\times\langle 3,4,0\rangle\right)=\langle 1,0,2\rangle\times\langle -4,3,-3\rangle=$
$=\left\langle 0\cdot\left(-3\right)-2\cdot 3,2\cdot\left(-4\right)-1\cdot\left(-3\right),1\cdot 3-0\cdot\left(-4\right)\right\rangle=\langle -6,-5,3\rangle$

d. $\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}=\left(\langle 1,0,2\rangle\cdot\langle 3,4,0\rangle\right)\langle 0,1,1\rangle-\left(\langle 1,0,2\rangle\cdot\langle 0,1,1\rangle\right)\langle 3,4,0\rangle=$
$=\left(1\cdot 3+0\cdot 4+2\cdot 0\right)\langle 0,1,1\rangle-\left(1\cdot 0+0\cdot 1+1\cdot 2\right)\langle 3,4,0\rangle$
$=3\langle 0,1,1\rangle-2\langle 3,4,0\rangle=\langle 0,3,3\rangle-\langle 6,8,0\rangle=\langle -6,-5,3\rangle$