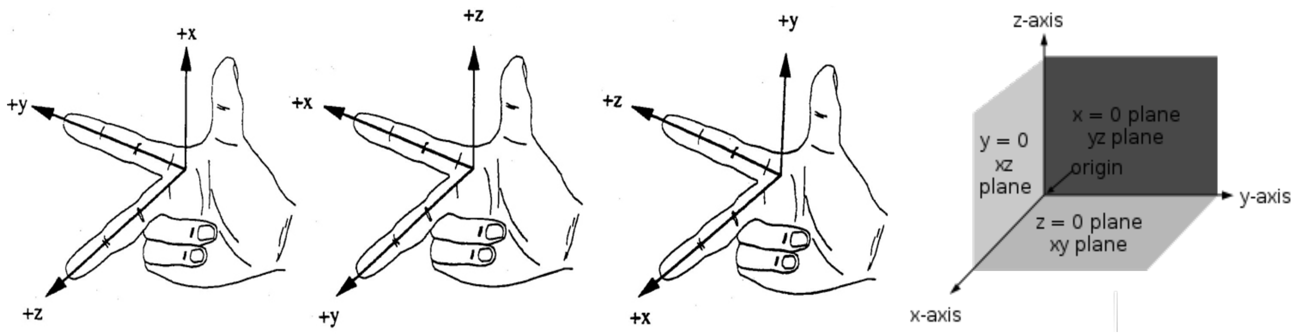
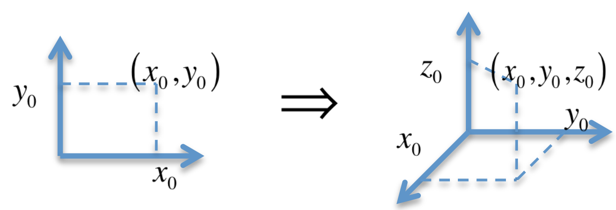
2 Vectors and the geometry of Space

2.1 Three dimensional Coordinate System (9.1)

Righ-handed coordinate system





Def: Plane in 3D is given by the formula $Ax + By + Cz = D$

Ex 1. The xy plane is given by

$$\{(x,y,z) \mid x,y \in \mathbb{R}, z = 0\} = \{(x,y,z) \mid z = 0\} = \{(x,y,0) \mid x,y \in \mathbb{R}\},$$

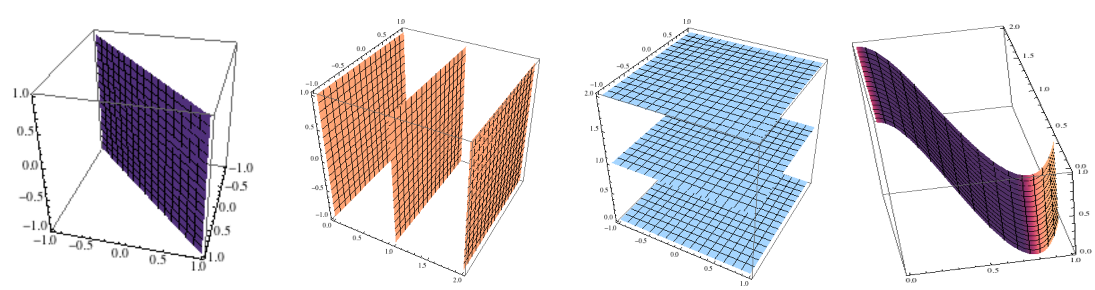
or $(x,y,0)$ or simply $z = 0$. The last representation has the form of the definition $Ax + By + Cz = 0 \cdot x + 0 \cdot y + 1 \cdot z = 0 = D$

Ex 2. A plane parallel to xy plane may be denoted as (x,y,z_0) or $z = z_0$

Ex 3. Similarly $y = y_0$ and $x = x_0$ are xz and yz planes respectively

Ex 4. A plane $x = y$, i.e. $Ax + By + Cz = 1 \cdot x - 1 \cdot y + 0 \cdot z = 0 = D$

Ex 5. A surface $y = 1 + \cos 4x$



Def: Euclidian Distance formula between points $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$ in space (3D) denoted as $|P_1P_2|$ and given as $|P_1P_2|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Compare to the Euclidian Distance formula between points $P_1=(x_1,y_1)$ and $P_2=(x_2,y_2)$ in plane (2D) that are given as $|P_1P_2|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

Ex 6. Show that midpoint between $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$ is given by $P_3\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$ by showing that $|P_1P_3|=|P_3P_2|$

$$|P_1P_3|=\sqrt{\left(\frac{x_1+x_2}{2}-x_1\right)^2+\left(\frac{y_1+y_2}{2}-y_1\right)^2+\left(\frac{z_1+z_2}{2}-z_1\right)^2}=\sqrt{\left(\frac{x_2}{2}-\frac{x_1}{2}\right)^2+\left(\frac{y_2}{2}-\frac{y_1}{2}\right)^2+\left(\frac{z_2}{2}-\frac{z_1}{2}\right)^2}$$

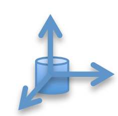
$$|P_3P_2|=\sqrt{\left(x_2-\frac{x_1+x_2}{2}\right)^2+\left(y_2-\frac{y_1+y_2}{2}\right)^2+\left(z_2-\frac{z_1+z_2}{2}\right)^2}=\sqrt{\left(\frac{x_2}{2}-\frac{x_1}{2}\right)^2+\left(\frac{y_2}{2}-\frac{y_1}{2}\right)^2+\left(\frac{z_2}{2}-\frac{z_1}{2}\right)^2}$$

Def: A sphere centered at 0 is given by formula $x^2+y^2+z^2=r^2$. If it centered at $P_1=(x_1,y_1,z_1)$, then it is given by $(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2$. Compare to the equation of circle centered at 0 $x^2+y^2=r^2$ or $(x-x_1)^2+(y-y_1)^2=r^2$ centered at $P_1=(x_1,y_1)$.

Ex 7. Describe $x^2+y^2+z^2\le r^2$: it is inside the sphere (equal give only surface).

Ex 8. Describe $0\le x,y,z\le 1$: a cube with side of size 1 and a corner at origin

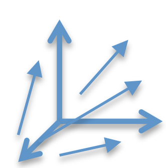
Ex 9. Give a geometric description to $\{(x,y,z)\mid x^2+y^2=r^2\}$: cylinder



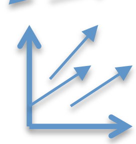
2.2 Vectors (9.2)

Microsoft Mathematics: a software for 3D visualization – useful tool
http://www.microsoft.com/en-us/download/details.aspx?id=15702

Def: Vector is mathematical object that specify length/magnitude and direction. We denote a vector by putting a little over on the top, or sometime in books by using bold font. $\vec{a},\vec{u},\vec{v}$ or **a**, **u**, **v**. We write a vector as a list of numbers in angle brackets $\langle x,y\rangle$ or $\langle x,y,z\rangle$.



A vector from point A to point B is denoted as $\overrightarrow{AB}$ and sketched as a line segment with an arrow points in the direction of the vector. However, the vector considered as a numerical quantity that doesn't represent the locations of A and B, but the location of point B relative to A as if A were the coordinate origin. Thus, vectors are position-less and therefore:



Def: two vectors considered equivalent if their magnitude\length and direction is equal, regardless the position. Similarly $\langle x_1,y_1,z_1\rangle=\langle x_2,y_2,z_2\rangle$ $\langle x_1,y_1\rangle=\langle x_2,y_2\rangle$ iff $x_1=x_2, y_1=y_2, z_1=z_2$.

Def: A vector from point $A(x_1,y_1,z_1)$ ($\tilde{A}(x_1,y_2)$) to point $B(x_2,y_2,z_2)$ ($\tilde{B}(x_2,y_2)$)is denoted as $\overrightarrow{AB}=\langle x_2-x_1,y_2-y_1,z_2-z_1\rangle$ ($\overrightarrow{\tilde{A}\tilde{B}}=\langle x_2-x_1,y_2-y_1\rangle$)

Ex 1. Displacement vector: Dv= end point – start point, |dv| = distance

Ex 2. Velocity vector: Vv = <velocity in x direction, velocity in y direction>, |Vv| = speed (in distance per time, like mph)

Ex 3. Force vector: $F=\langle F_1,F_2,F_3\rangle$, $|F|$ is a total force (in Newtons).

2.2.1 Vector Arithmetic

Let $\vec{a}=\langle a_1,a_2\rangle,\vec{b}=\langle b_1,b_2\rangle,\vec{c}=\langle c_1,c_2,c_3\rangle,\vec{d}=\langle d_1,d_2,d_3\rangle$ be a vectors and *k* a constant. In order to distinct vectors and numbers we will call *k* a **scalar**.

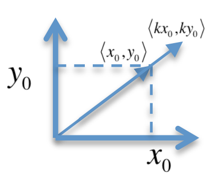
Def: The length or a magnitude of 2D, 3D vectors $\vec{a},\vec{c}$ is defined by $|\vec{a}|=\sqrt{a_1^2+a_2^2}$ and $|\vec{c}|=\sqrt{c_1^2+c_2^2+c_3^2}$ respectively (don't confuse with absolute value).

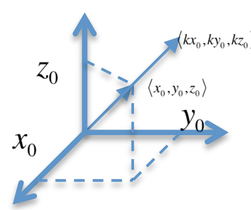
Def: The **unit vector** is a vector with magnitude\length equal 1.

Def: Scalar Multiplication defined as $ka=\langle ka_1,ka_2\rangle$ and $kc=\langle kc_1,kc_2,kc_3\rangle$. If k<0 then the vector change direction. Note also that $|k\vec{a}|=\sqrt{k^2a_1^2+k^2a_2^2}=|k|\sqrt{a_1^2+a_2^2}=|k||\vec{a}|$ and

$|k\vec{c}|=\sqrt{k^2c_1^2+k^2c_2^2+k^2c_3^2}=|k|\sqrt{c_1^2+c_2^2+c_3^2}=|k||\vec{c}|$

Thus scalar multiplication scales (either stretch or squeeze) vector magnitude\length.

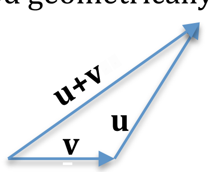


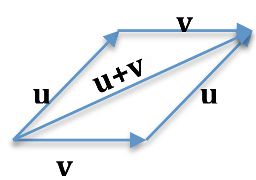


Note: Every vector **u** can be converted into a unit vector by multiplying it by very special scalar equal to reciprocal of it's magnitude: $\left|\frac{1}{|\vec{u}|}\vec{u}\right|=\left|\frac{\vec{u}}{|\vec{u}|}\right|=\frac{|\vec{u}|}{|\vec{u}|}=1$.

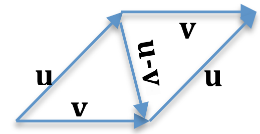
Ex 4. $\langle 1,2,3\rangle\Rightarrow\frac{1}{|\langle 1,2,3\rangle|}\langle 1,2,3\rangle=\frac{1}{\sqrt{1+2^2+3^2}}\langle 1,2,3\rangle=\frac{1}{14}\langle 1,2,3\rangle=\left\langle\frac{1}{14},\frac{1}{7},\frac{3}{14}\right\rangle$

Def: **Vector Addition** defined as $\vec{a}+\vec{b}=\langle a_1+b_1,a_2+b_2\rangle$ and $\vec{c}+\vec{d}=\langle c_1+d_1,c_2+d_2,c_3+d_3\rangle$. The addition of vectors can be understood geometrically as as a Triangle Law or a Parallelogram Law, which are illustrated below.





Def: **Vector Subtraction** defined by $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$ where $-\vec{v}$ is the vector with the same magnitude but opposite direction $\vec{v}$.



Properties of vectors:

1. $\vec{a}+\vec{b}=\langle a_1+b_1,a_2+b_2\rangle=\langle b_1+a_1,b_2+a_2\rangle=\vec{b}+\vec{a}$
2. $(\vec{a}+\vec{b})+\vec{c}=\langle a_1+b_1,a_2+b_2\rangle+\vec{c}=\langle a_1+b_1+c_1,a_2+b_2+c_2\rangle=$
   $=\vec{a}+\langle b_1+c_1,b_2+c_2\rangle=\vec{a}+(\vec{b}+\vec{c})$
3. $\vec{a}+\vec{0}=\langle a_1+0,a_2+0\rangle=\langle a_1,a_2\rangle=\vec{a}$
4. $\vec{a}+(-\vec{a})=\langle a_1+(-a_1),a_2+(-a_2)\rangle=\langle 0,0\rangle=\vec{0}$
5. $k(\vec{a}+\vec{b})=\langle k(a_1+b_1),k(a_2+b_2)\rangle=\langle ka_1,ka_2\rangle+\langle kb_1,kb_2\rangle=k\vec{a}+k\vec{b}$
6. $(k+m)\vec{a}=\langle (k+m)a_1,(k+m)a_2\rangle=\langle ka_1+ma_2,ka_2+ma_2\rangle=$
   $=\langle ka_1,ka_2\rangle+\langle ma_1,mba_2\rangle=k\vec{a}+m\vec{a}$
7. $(km)\vec{a}=\langle kma_1,kma_2\rangle=k\langle ma_1,ma_2\rangle=k(m\vec{a})$

2.2.2 Standard basis (i-j-k notations)

We define 3 very important vectors, called a **Standard Basis:** $\mathbf{i}=\langle 1,0,0\rangle$; $\mathbf{j}=\langle 0,1,0\rangle$; $\mathbf{k}\langle 0,0,1\rangle$ in 3D or equivalently $\mathbf{i}=\langle 1,0\rangle$; $\mathbf{j}=\langle 0,1\rangle$ in 2D.

Thm: Every vector can be expressed as a sum of vectors of the standard basis as following

$$\langle a_1,a_2,a_3\rangle=a_1\langle 1,0,0\rangle+a_2\langle 0,1,0\rangle+a_3\langle 0,0,1\rangle=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$$
$$\langle a_1,a_2\rangle=a_1\langle 1,0\rangle+a_2\langle 0,1\rangle=a_1\mathbf{i}+a_2\mathbf{j}$$