1.6 Taylor and Maclaurin Series (8.7)

We look to represent more functions as series. Consider that a function f has a representation as power series in some interval $|x-a|<R$. If so we looking for a way to find the coefficients of the expansion

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\ldots .$$

We note that $f(a)=c_0$, furthermore

$f'(x)=\sum_{n=0}^{\infty}nc_n(x-a)^{n-1}=\sum_{n=1}^{\infty}nc_n(x-a)^{n-1}=c_1+2c_2(x-a)+3c_3(x-a)^2+\ldots$, and therefore $f'(a)=c_1$,

$f''(x)=2c_2+2\cdot 3c_3(x-a)+3\cdot 4c_4(x-a)^2+\ldots$, and therefore $f''(a)=2c_2$ and so on:

$$f^{(n)}(a)=2\cdot 3\cdot 4\cdots nc_n=n!c_n\Rightarrow c_n=\frac{f^{(n)}(a)}{n!}$$

Thm: If a function f has a power series representation (expansion) about a, that is $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$, then $c_n=\frac{f^{(n)}(a)}{n!}$, i.e. $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$.

Def: the series $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ are called **Taylor series/expansion** of the function f (about/centered at) at a. For a special case a=0 the series $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ called **Maclaurin series**.

Ex 1. Find Maclaurin (Taylor about 0) expansion for $e^x: f^{(n)}(x)=e^x,\forall n\geq 0$ therefore

$$e^x=e^0+e^0(x-0)+\frac{e^0}{2}(x-0)^2+\frac{e^0}{6}(x-0)^3+\ldots=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!},\ R=\infty$$

Ex 2. Find Maclaurin (Taylor about 0) expansion for $\sin x$:

$$f(x)=\sin x,\quad f'(x)=\cos x,\quad f''(x)=-\sin x,\qquad f'''(x)=-\cos x$$

$$f^{(n)}(x)=\begin{cases}(-1)^m\sin(x) & n=2m\\ (-1)^m\cos(x) & n=2m+1\end{cases},\ m=0,1,2,3\ldots$$

$$\Rightarrow f^{(n)}(0)=\begin{cases}0 & n=2m\\ (-1)^m & n=2m+1\end{cases},\ m=0,1,2,3\ldots$$

therefore

$$\sin x=0+\frac{1}{1}x+\frac{0}{2}x^2-\frac{1}{6}x^3+\frac{0}{24}x^4+\frac{1}{5!}x^5+\frac{0}{6!}x^6-\frac{1}{7!}x^7+\ldots=$$

$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!},\ R=\infty$$

Ex 3. Find Maclaurin (Taylor about 0) expansion for $\cos x$:

$$f(x)=\cos x,\quad f'(x)=-\sin x, f''(x)=-\cos x,\qquad f'''(x)=\sin x$$

$$f^{(n)}(x)=\begin{cases}(-1)^m\cos(x) & n=2m\\ (-1)^{m+1}\sin(x) & n=2m+1\end{cases}, m=0,1,2,3...$$

$$\Rightarrow f^{(n)}(0)=\begin{cases}(-1)^m & n=2m\\ 0 & n=2m+1\end{cases}, m=0,1,2,3...$$

therefore

$$\cos x=1+\frac{1}{1}x-\frac{1}{2}x^2+\frac{0}{6}x^3+\frac{1}{24}x^4+\frac{0}{5!}x^5-\frac{1}{6!}x^6+\frac{0}{7!}x^7+...=$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}, R=\infty$$

Another way $\cos x=\frac{d}{dx}\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{d}{dx}\frac{x^{2k+1}}{(2k+1)!}=\sum_{k=0}^{\infty}(-1)^k\frac{(2k+1)x^{2k}}{(2k+1)!}=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}, R=\infty$

Ex 4. $x\cos x=x\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k)!}, R=\infty$

$$f(x)=\frac{\sin x}{x}=\frac{1}{x}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k+1)!}, R=\infty$$

Ex 5. Find Taylor series for $\frac{1}{x}$ about 1:

$$f(x)=\frac{1}{x}=1;\quad f'=-\frac{1}{x^2}=-1; f''=\frac{2}{x^3}=2;\quad f^3=-\frac{6}{x^4}=-6;\qquad f^4=\frac{24}{x^5}=4!;\ f^{(n)}(1)=(-1)^n n!$$

$$\frac{1}{x}=\sum_{n=0}^{\infty}(-1)^n(x-1)^n=1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-(x-1)^5...$$

Def: Taylor\Maclaurin polynomial is a partial sum of Taylor\Maclaurin series, i.e.

$T_m=\sum_{n=0}^{m}\frac{f^{(n)}(0)}{n!}x^n$. The remainder is defined as $R_m(x)=f(x)-T_m(x)$

Thm: If $f(x)=T_m(x)+R_m(x)$ and $\lim_{m\to\infty}R_m(x)=0,\forall x\in(a-R,a+R)$ then

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n,\forall x\in(a-R,a+R).$$

Furthermore, if $\max_{|x-a|<d}\left|f^{(m+1)}(x)\right|=M$ (i.e. $\left|f^{(m+1)}(x)\right|\le M,\forall|x-a|<d$), where $d>0$ is small positive number then $\left|R_m(x)\right|\le\frac{M}{(m+1)!}|x-a|^{m+1},\forall|x-a|<d$

Ex 6. Approximate $\sqrt{2}$ using 3[rd] order Taylor polynomial about 1, bound the remainder:

$$f(x)=\sqrt{x}=1;\ f'(x)=\frac{1}{2\sqrt{x}}=\frac{1}{2};\qquad f''(x)=-\frac{1}{4x^{3/2}}=-\frac{1}{4};f'''(x)=\frac{3}{8x^{5/2}}=\frac{3}{8}$$

$$1.41421=\sqrt{2}\approx 1+\frac{1}{2}(2-1)-\frac{1}{4}\frac{(2-1)^2}{2}+\frac{3}{8}\frac{(2-1)^3}{6}=1+\frac{1}{2}-\frac{1}{8}+\frac{3}{48}=\frac{23}{16}=1.4375$$

$$\left|R_m(2)\right|\le\frac{\max_{x\in(1,2)}\left|f^{(4)}(x)\right|}{4!}\left|2-1\right|^4=\frac{\max_{x\in(1,2)}\left|-\frac{15}{16x^{7/2}}\right|}{24}=\frac{5}{16\cdot 8}=\frac{5}{128}\approx 0.039$$

$$err=\left|1.4375 - 1.41421\right|=0.02329<0.039$$

Ex 7. Approximate $\ln\frac{1}{2}$ using order Taylor polynomial about 1:

$$f(x)=\ln x=0;\qquad f'=\frac{1}{x}=1;\qquad f''(x)=-\frac{1}{x^2}=-1;\qquad f'''(x)=\frac{2}{x^3}=2;$$

$$-0.693\approx\ln\frac{1}{2}=0+\left(\frac{1}{2}-1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)^2+\frac{2}{6}\left(\frac{1}{2}-1\right)^3=-\frac{1}{2}-\frac{1}{8}-\frac{1}{24}=-\frac{2}{3}\approx-0.666$$

$$\left|R_m\left(\frac{1}{2}\right)\right|\le\frac{\max_{x\in(0.5,1)}\left|f^{(4)}(x)\right|}{4!}\left|\frac{1}{2}-1\right|^4=\frac{\max_{x\in(0.5,1)}\left|-\frac{6}{x^4}\right|}{24}\frac{1}{2^4}=\frac{\frac{6}{1/2^4}}{24}\frac{1}{2^4}=\frac{6}{24}=\frac{1}{4}=0.25$$

$$err=\left|0.693 - 0.666\right|=0.027<0.25$$

Ex 8. Find Maclaurin series for $f(x)=(1+x)^k$

$$f(x)=(1+x)^k\Big|_{x=0}=1;\ f'=k(1+x)^{k-1}\Big|_{x=0}=k;f''=k(k-1)(1+x)^{k-2}\Big|_{x=0}=k(k-1)$$

$$f^{(n)}(0)=k(k-1)(k-2)\cdots(k-n+1)$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n=\sum_{n=0}^{\infty}\binom{k}{n}x^n$$

Def: The numbers $\binom{k}{n}=\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$ called binomial coefficients.

Ex 9.

$$\frac{1}{\sqrt[3]{36-x}}=\frac{1}{3\sqrt[3]{\left(1-\frac{x}{27}\right)}}=\frac{1}{3}\left(1-\frac{x}{27}\right)^{-1/3}=\frac{1}{3}\sum_{n=0}^{\infty}\binom{-\frac{1}{3}}{n}\left(-\frac{x}{27}\right)^{n}=$$

$$=\frac{1}{3}\left(1+\left(-\frac{1}{3}\right)\left(-\frac{x}{27}\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}\left(-\frac{x}{27}\right)^{2}+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{4}{3}-1\right)}{3!}\left(-\frac{x}{27}\right)^{3}+\cdots\right.$$

$$\left.+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{4}{3}-1\right)\cdots\left(-\frac{1}{3}-n+1\right)}{n!}\left(-\frac{x}{27}\right)^{n}\right)$$

Ex 10. Neat example

$$\cos x+i\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}+i\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}=$$

$$=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right)=$$

$$=1+ix-\frac{x^2}{2!}-i\frac{x^3}{3!}+\frac{x^4}{4!}+i\frac{x^5}{5!}-\frac{x^6}{6!}-i\frac{x^7}{7!}+\ldots=$$

$$=1+ix+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\frac{(ix)^4}{4!}+\frac{(ix)^5}{5!}+\frac{(ix)^6}{6!}+\frac{(ix)^7}{7!}+\ldots=\sum_{k=0}^{\infty}\frac{(ix)^k}{k!}=e^{ix}$$

$i=\sqrt{-1};\quad i^2=-1;\quad i^3=i^2i=-i;\quad i^4=i^2i^2=(-1)(-1)=1$

$i^5=i^4i=i;\quad i^6=i^4i^2=-1;\quad i^7=i^6i=-i$