1.5 Representation of Functions as Power Series (8.6)

Reminder: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, |r|<1$

Ex 1. $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

Ex 2. $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 \ldots$

Ex 3. $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 \ldots$

Ex 4. $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}$

Ex 5. $\frac{x}{1-x^2} = x\sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} x^{2n+1}$

Ex 6. $\frac{x}{2^2 - x^2} = \frac{1}{2} \cdot \frac{x/2}{1 - x^2/2^2} = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^{2n+2}}$

Thm: If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence R>0, then the function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is continues, differentiable and integrable on the interval $(a-R, a+R)$ and

$$f'(x) = \frac{d}{dx}\sum_{n=1}^{\infty} c_n (x-a)^n = \sum_{n=1}^{\infty} \frac{d}{dx} c_n (x-a)^n = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int f(x)dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

with the same radius of convergence R. However the convergence at end points may change.
Note: The theorem above state that the power series can be integrated/differentiated term-by-term. However, for other types of series of functions ($\sum_{n=0}^{\infty} c_n g_n(x)$) the situation is not as simple.

Differentiation:

Ex 7. $\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1}$

Ex 8.

Ex 9. $\frac{1}{(1+x)^2} = -\frac{d}{dx}\frac{1}{1+x} = -\frac{d}{dx}\sum_{n=0}^{\infty} (-1)^n x^n = -\sum_{n=0}^{\infty} (-1)^n \frac{d}{dx} x^n = -\sum_{n=0}^{\infty} (-1)^n n x^{n-1} =$

$$= \sum_{n=1}^{\infty} -(-1)^n n x^{n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$$

Ex 10. $$\frac{-2x}{\left(1+x^2\right)^2}=\frac{d}{dx}\frac{1}{1+x^2}=\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^{2n}=\sum_{n=0}^{\infty}(-1)^n\frac{d}{dx}x^{2n}=$$
$$=\sum_{n=0}^{\infty}(-1)^n 2nx^{2n-1}=\sum_{n=1}^{\infty}(-1)^n 2nx^{2n-1}$$

Ex 11. $$\frac{2x}{\left(1-x^2\right)^2}=\frac{d}{dx}\frac{1}{1-x^2}=\frac{d}{dx}\sum_{n=0}^{\infty}x^{2n}=\sum_{n=0}^{\infty}\frac{d}{dx}x^{2n}=\sum_{n=0}^{\infty}2nx^{2n-1}=\sum_{n=1}^{\infty}2nx^{2n-1}$$

Ex 12. $$\frac{1+x^2}{\left(x^2-1\right)^2}=\frac{d}{dx}\frac{x}{1-x^2}=\frac{d}{dx}\sum_{n=0}^{\infty}x^{2n+1}=\sum_{n=0}^{\infty}\frac{d}{dx}x^{2n+1}=\sum_{n=0}^{\infty}(2n+1)x^{2n}$$

Ex 13. $$\frac{3^2+x^2}{\left(3^2-x^2\right)^2}=\frac{3^2-x^2+2x^2}{\left(3^2-x^2\right)^2}=\frac{d}{dx}\frac{x}{3^2-x^2}=\frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{3^{2n+2}}=\sum_{n=0}^{\infty}\frac{d}{dx}\frac{x^{2n+1}}{3^{2n+2}}=\sum_{n=0}^{\infty}(2n+1)\frac{x^{2n}}{3^{2n+2}}$$

Integration:

Ex 14. $$-\ln(1-x)=\int\frac{dx}{1-x}=\int\sum_{n=0}^{\infty}x^n dx=\sum_{n=0}^{\infty}\int x^n\,dx=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}+C=\sum_{n=1}^{\infty}\frac{x^n}{n}+C$$

Ex 15. $$\ln(x+1)=\int\frac{dx}{x+1}=\int\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)dx=\sum_{n=0}^{\infty}\int(-1)^n x^n\,dx=$$
$$=\sum_{n=0}^{\infty}(-1)^n\int x^n\,dx=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}+C=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}+C$$

Ex 16. $$\arctan x=\int\frac{dx}{1+x^2}=\int\sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx=\sum_{n=0}^{\infty}(-1)^n\int x^{2n}\,dx=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}+C$$

Ex 17. $$\operatorname{arctanh}x=\frac{1}{2}\ln\frac{1+x}{1-x}=\int\frac{dx}{1-x^2}=\int\sum_{n=0}^{\infty}x^{2n}dx=\sum_{n=0}^{\infty}\int x^{2n}\,dx=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}+C$$

Ex 18. $$-\frac{1}{2}\ln\left(1-x^2\right)=\int\frac{x}{1-x^2}dx=\int\sum_{n=0}^{\infty}x^{2n+1}\,dx=\sum_{n=0}^{\infty}\int x^{2n+1}\,dx=\sum_{n=0}^{\infty}\frac{x^{2n+2}}{2n+2}+C$$

Ex 19. $$-\frac{1}{2}\ln\left(25-x^2\right)=\int\frac{x}{5^2-x^2}dx=\int\sum_{n=0}^{\infty}\frac{x^{2n+1}}{5^{2n+2}}dx=\sum_{n=0}^{\infty}\int\frac{x^{2n+1}}{5^{2n+2}}dx=\sum_{n=0}^{\infty}\frac{1}{2n+2}\frac{x^{2n+2}}{5^{2n+2}}+C$$