

## 1.4 Power Series

**Def:** A power series is a series of the following form  $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$  where  $x$  is variables and  $c_n$  are constants called coefficients.

The power series may converge or diverge. The sum of series is a function  $f(x) = \sum_{n=0}^{\infty} c_n x^n$  whose domain is the set of  $x$ 's for which the series converges.

Ex 1. For  $c_n = 1, \forall n \in \mathbb{N}$  one get geometric series  $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$  which converges for  $-1 < x < 1$  and diverges  $|x| \geq 1$ , since  $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{x^n} \right| = |x|$ .

**Def:** More general power series are  $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \dots$  these series are centered at  $a$ , also said to be about  $a$ .

**Thm:** For a given power series  $\sum_{n=0}^{\infty} c_n (x-a)^n$  there is are only three possibilities:

- I. The series converges only when  $x = a$
- II. The series converges for all  $x$
- III. There is a positive number  $R$ , such that the series converges if  $|x-a| < R$  and diverges  $|x-a| > R$ .

**Note:** The number  $R$  is **called a radius of convergence**. By convention in case (1) we will say  $R=0$ , in case (2)  $R = \infty$ . In the 3 case the theorem talking only about  $a - R < x < a + R$  but nothing about end points  $x = a \pm R$ . At end points all options are open, i.e. it may converge or diverge at end points depends on particular case.

To determine the radius of convergence we use the Ratio test, however it doesn't work for the endpoints. Therefore the endpoints must be checked with another test.

Ex 2.  $\sum_{n=0}^{\infty} n! x^n \Rightarrow \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = |(n+1)x| \rightarrow \infty, \forall x \neq 0$ , thus it converges only when  $x = 0$ , i.e.  $R=0$ .

Ex 3.  $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2} \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x-5)^{n+1}}{(n+1)^2} \frac{n^2}{(x-5)^n} \right| = \left| (x-5) \left( \frac{n}{n+1} \right)^2 \right| = \left| (x-5) \left( \frac{1}{1+1/n} \right)^2 \right| \rightarrow |x-5|$ , thus

it is convergent when  $|x-5| < 1 \Rightarrow -1 < x-5 < 1 \Rightarrow 4 < x < 6 \Rightarrow R=1$ . The end points we

check separately: 1)  $\sum_{n=1}^{\infty} \frac{(4-5)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$  and 2)  $\sum_{n=1}^{\infty} \frac{(6-5)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$  and find that the series converges for both, i.e. the series converges for  $|x-5| \leq 1$ .

$$\text{Ex 4. } \sum_{n=1}^{\infty} \frac{n^3(x+7)^n}{5^{n-1}} \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^3(x+7)^{n+1}}{5^{(n+1)-1}} \frac{5^{n-1}}{n^3(x+7)^n} \right| = \left| \left( \frac{n+1}{n} \right)^3 \frac{x+7}{5} \right| = \frac{1}{5} \left( 1 + \frac{1}{n} \right)^3 |x+7| \quad ,$$

thus the series converges when  $|x+7| < 5$ , i.e.  $-5 < x+7 < 5 \Rightarrow -12 < x < -2 \Rightarrow R = 5$ . At the end points, the series diverges:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^3(-2+7)^n}{5^{n-1}} &= \sum_{n=1}^{\infty} \frac{n^3 5^n}{5^{n-1}} = 5 \sum_{n=1}^{\infty} n^3 \\ \sum_{n=1}^{\infty} \frac{n^3(-12+7)^n}{5^{n-1}} &= \sum_{n=1}^{\infty} \frac{n^3(-5)^n}{5^{n-1}} = \sum_{n=1}^{\infty} (-1)^n \frac{n^3 5^n}{5^{n-1}} = 5 \sum_{n=1}^{\infty} (-1)^n n^3 \end{aligned}$$