1.3.4 Absolute convergence

Def: A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

Thm: If a series $\sum a_n$ is absolutely convergent then it is convergent.

It is true because 1) $0 \le a_n + |a_n| \le 2|a_n|$, 2) $\sum |a_n|$ is convergent and so $2\sum |a_n|$ and by comparison test ($\sum (a_n + |a_n|) \le 2\sum |a_n|$) also $\sum a_n + |a_n|$ is convergent. Finally $\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$

Def: A series $\sum a_n$ is called conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Ex 1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} ...$ is absolutely convergent since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Ex 2. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} ...$ is convergent due to alternating series theorem: 1) $b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$ and 2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

However, it is not absolutely convergent since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

1.3.5 The Ratio Test

The following test is very useful in determining absolute convergence.

The ratio test theorem:

- If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ then the series $\sum a_n$ is absolutely convergent (and therefore convergent)
- If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ (including $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$) then the series $\sum a_n$ divergent.
- If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ then the ratio test inconclusive, that is we have to use another method to determine convergence or divergence of $\sum a_n$.

Ex 3. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1/(n+1)^p}{1/n^p} = \lim_{n\to\infty}\frac{n^p}{(n+1)^p} = \left(\lim_{n\to\infty}\frac{n}{n+1}\right)^p \to 1$, thus the ratio test is inconclusive in case of p-series. Indeed we know that it may converge or diverge depends of value of p, while the ratio test is independent of that value.

Ex 4. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)^5/7^{n+1}}{(-1)^n n^5/7^n}\right| = \frac{7^n}{7^{n+1}}\left(\frac{n+1}{n}\right)^5 = \frac{1}{7}\left(1+\frac{1}{n}\right)^5 \to \frac{1}{7} < 0$, thus $\sum(-1)^n\frac{n^5}{7^n}$ converges.

Ex 5. $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = \frac{\cancel{n!}\,\cancel{(n+1)}}{\cancel{(n+1)}\,(n+1)^n}\cdot\frac{n^n}{\cancel{n!}} = \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n} = \left(\left(1+\frac{1}{n}\right)^n\right)^{-1} \to \frac{1}{e} < 1$

thus $\sum\frac{n!}{n^n}$ converges. Similarly, $\sum\frac{n^n}{n!}$ diverges, since $\left|\frac{a_{n+1}}{a_n}\right| \to e > 1$.

1.3.6 Estimating Sums

1.3.6.1 Remainder Estimate for the Integral Test

Thm: Let $a_k = f(k)$, where *f*(*x*) is a continues, positive, decreasing function for $x \ge n$ (these are conditions for a integral test). Consider that $S = \sum_{n=1}^{\infty} a_n$ is convergent series. Let $S_m = \sum_{n=1}^{m} a_n$ be a partial sum and $R_m = S - S_m = \sum_{n=m+1}^{\infty} a_n$ is the remainder, then $\int_{m+1}^{\infty} f(x) \le R_m \le \int_{m}^{\infty} f(x)$.

Ex 6. a) How good S_3 estimates $S = \sum_{n=1}^{\infty}\frac{1}{n^2+1}$?

$$\sum_{n=1}^{3}\frac{1}{n^2+1} = \frac{1}{1+1}+\frac{1}{4+1}+\frac{1}{9+1} = \frac{1}{2}+\frac{1}{5}+\frac{1}{10} = \frac{5+2+1}{10} = \frac{8}{10}$$

By the theorem above

$$R_3 \le \int_3^{\infty}\frac{dx}{x^2+1} = \lim_{t\to\infty}\int_3^t\frac{dx}{x^2+1} = \lim_{t\to\infty}\arctan x\Big|_3^t = \lim_{t\to\infty}(\arctan t - \arctan 3) = \frac{\pi}{2} - \arctan 3 \approx 0.321$$

so the estimation isn't that good.

b) Can we improve the estimation without calculating more terms?
According to estimating theorem

$$\int_{m+1}^{\infty} f(x) \le R_m = S - S_m \le \int_m^{\infty} f(x) \Rightarrow S_m + \int_{m+1}^{\infty} f(x) \le S \le S_m + \int_m^{\infty} f(x)$$

so $1.04498 \approx \frac{8}{10}+\frac{\pi}{2}-\arctan 4 = \frac{8}{10}+\int_{m+1}^{\infty} f(x) \le S \le \frac{8}{10}+\int_m^{\infty} f(x) = \frac{8}{10}+\frac{\pi}{2}-\arctan 3 \approx 1.12175$

One can take a midpoint of this approximation, to get $S \approx 1.08336$. The more precise sum is $S = 1.07667$, so it is clear that we improved the approximation (for $R_m \approx -6.69 \times 10^{-3}$).

c) How many terms do we need in order to make $R_n < 10^{-3}$.

$$R_m \le \int_m^\infty \frac{dx}{x^2+1} = \lim_{t\to\infty} \int_m^t \frac{dx}{x^2+1} = \lim_{t\to\infty} \arctan x\Big|_m^t = \lim_{t\to\infty}\left(\arctan t - \arctan m\right) = \frac{\pi}{2} - \arctan m$$

so we look for m such that

$$\frac{\pi}{2} - \arctan m < 10^{-3} \Rightarrow \arctan m > \frac{\pi}{2} - 10^{-3} \Rightarrow m > \tan\left(\frac{\pi}{2} - 10^{-3}\right) = 1000$$

1.3.6.2 Alternating Estimation Theorem

Thm: If an alternating series $S = \sum b_n = \sum (-1)^n a_n$ satisfy $a_{n+1} \le a_n$ and $\lim_{n\to\infty} a_n = 0$ then $|R_m| = |S - S_m| \le b_n$

Ex 7. How many terms are needed to approximate $\sum (-1)^n \frac{1}{n^3}$ so that the approximation is accurate to within 10^{-3}. Answer $\frac{1}{n^3} < 10^{-3} \Rightarrow n^3 > 10^3 \Rightarrow n > 10$