1.3 Convergence Tests + Estimating Sums (8.3-8.4)

When a sequence of partial sum $\{S_n\}$ of a series has a simple formula it is easy to find the sum, however this is not always the case. We will learn several convergence tests that doesn't require evaluating a sum. In some cases these methods lead to estimating of the sum

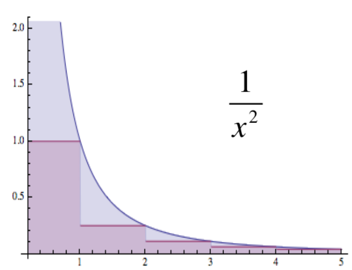
1.3.1 The Integral test

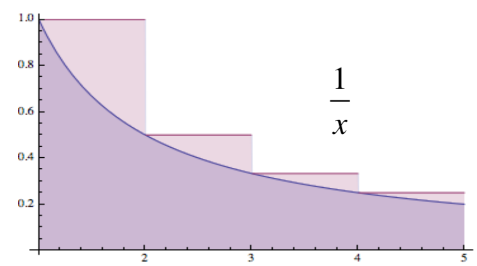
Theorem: Let f be continues, positive decreasing function on $[m,\infty)$ and let $a_n = f(n)$ for $n \ge m$ then the series $\sum_{n=m}^{\infty} a_n$ converges if and only if $\int_m^{\infty} f(x)dx$.

Note: Since finite number of terms cannot affect convergence of infinite series, it is enough that $f(x)$ decreasing on $[M,\infty)$ where $M > m$.

The following graphical examples intuitively explain the theorem.

- The graph of the area under $\frac{1}{x^2}$ is a convergent integral. This area bound the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$; therefore convergent (to $\frac{\pi^2}{6}$, the proof is out of our scope).
- The area under $\frac{1}{x}$ is infinite, i.e. divergent improper integral; therefore the series $\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}dx$ are also divergent.





Ex 1. Determine whether the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges

Since we want to use integral test, we first show that $f(x) = xe^{-x^2}$ is positive and decreasing on $[1,\infty]$. The function is trivially positive, since both x and exponent are positive on $[1,\infty]$. To show it is decreasing function we will look at derivative

$f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2} = 0 \Leftrightarrow 1-2x^2 = 0 \Leftrightarrow x^2 = \frac{1}{2} \Leftrightarrow x = \pm\frac{1}{\sqrt{2}}$, neither in our scope, so no critical points in $[1,\infty]$, i.e. the first derivative doesn't change the sign in $[1,\infty]$. Therefore, we can check the sign of the first derivative at any $x > 1$, e.g. $f(2) < 0$. Thus, we can proceed with the integral test.

$$\int_1^\infty xe^{-x^2}\,dx = \int_1^\infty -\frac{1}{2}\frac{d}{dx}e^{-x^2}\,dx = -\frac{1}{2}\lim_{\lambda\to\infty}\int_1^\lambda \frac{d}{dx}e^{-x^2}\,dx = -\frac{1}{2}\lim_{\lambda\to\infty}\left(e^{-x^2}\right)_1^\lambda == -\frac{1}{2}\lim_{\lambda\to\infty}\left(e^{-\lambda^2}-e^{-1}\right) = \frac{1}{2e}$$

Therefore $\sum_{n=1}^\infty ne^{-n^2}$ converges.

Ex 2. For what values of p is the series $\sum_{n=1}^\infty \frac{1}{n^p}$ convergent?

For p<0: $\sum_{n=1}^\infty \frac{1}{n^p} = 1 + 2^{|p|} + 3^{|p|} + \ldots = \infty$, another method $\lim_{n\to\infty}\frac{1}{n^p} = \infty \neq 0$, any way it diverges.

For p=0: $\sum_{n=1}^\infty \frac{1}{n^p} = 1+1+1+\ldots = \infty$, another method $\lim_{n\to\infty}\frac{1}{n^p} = 1 \neq 0$, anyway it diverges.

If p>0, then $f(x) = \frac{1}{x^p}$ is positive and decreasing on $[1,\infty]$.

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty}\left.\frac{x^{-p+1}}{-p+1}\right|_1^t = \lim_{t\to\infty}\frac{t^{-p+1}-1}{p-1} = \frac{1}{p-1}\lim_{t\to\infty}\left(t^{1-p}-1\right) = \begin{cases} \text{convergent}\left(\dfrac{1}{1-p}\right) & p>1 \\ \text{divergent}\left(\dfrac{1}{"\infty"}\right) & 0<p\le 1 \end{cases}$$

1.3.2 Comparison Tests

1.3.2.1 The Comparison Test

Thm: Let $\sum a_n, \sum b_n$ be series with $a_n, b_n \ge 0$, then

- If $\sum b_n$ is convergent and $a_n \le b_n$ then $\sum a_n$ is also convergent
- If $\sum b_n$ is divergent and $a_n \ge b_n$ then $\sum a_n$ is also divergent

Ex 3. $\frac{1}{2^n+1} < \frac{1}{2^n} \Rightarrow \sum_{n=1}^\infty \frac{1}{2^n+1} < \sum_{n=1}^\infty \frac{1}{2^n} = 1$, thus $\frac{1}{2^n+1}$ convergent

Ex 4. $\frac{2n-1}{7n^3+5n^2+2} < \frac{2n}{7n^3} = \frac{2}{7n^2} \Rightarrow \sum_{n=1}^\infty \frac{2n}{7n^3} < \frac{2}{7}\sum_{n=1}^\infty \frac{1}{n^2} = \frac{2}{7}\frac{\pi^2}{6} = \frac{\pi^2}{21}$, thus $\frac{2n-1}{7n^3+5n^2+2}$ is convergent

Ex 5. $\frac{1}{\sqrt{n}} = \frac{\sqrt{n}}{n} \ge \frac{1}{n}$, thus $\frac{1}{\sqrt{n}}$ is divergent.

1.3.2.2 The Limit Comparison Test

Thm: Let $\sum a_n, \sum b_n$ be series with $a_n, b_n \ge 0$. If $\lim_{n\to\infty}\frac{a_n}{b_n} = c$ where $c>0$ is a finite constant, then either both series converge or both diverge.

Ex 6. Determine whether $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ is converges or diverges. We choose $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ for a limit comparison test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2}{n^3+1}}{1/n} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = \lim_{n\to\infty} \frac{1}{1+1/n^3} = 1$$

Thus, we got $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ is divergent.

1.3.3 Alternating Series

Def: Let $a_n > 0$, then $\begin{aligned} \sum_{n=1}^{\infty} b_n &= \sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - + \ldots \\ \sum_{n=1}^{\infty} c_n &= \sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + - \ldots \end{aligned}$ are alternating series.

Thm: If an alternating series, either $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - + \ldots$ or $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + - \ldots$, where $a_n > 0$ satisfy

1) $a_{n+1} \le a_n$ and
2) $\lim_{n\to\infty} a_n = 0$

Then the series are converges.

Ex 7. Test the following series for convergence $\sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k \ln k}$

Consider $f(x) = \frac{1}{x \ln x}$, one found $f'(x) = \frac{-\left(\ln x + x \cdot \frac{1}{x}\right)}{(x \ln x)^2} = \frac{-(\ln x + 1)}{(x \ln x)^2} < 0, \forall x \ge 2$ that f is decreasing function. Thus $a_{k+1} = \frac{1}{(k+1)\ln(k+1)} \le \frac{1}{k \ln k} = a_k$. Furthermore $\lim_{k\to\infty} = \frac{1}{k \ln k} = 0$, therefore the series $\sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k \ln k}$ converges.

Ex 8. The series $\sum_{n=1}^{\infty} (-1)^n \frac{7n^2}{3n^2-4}$ diverges since $\lim_{n\to\infty} \frac{7n^2}{3n^2-4} = \lim_{n\to\infty} \frac{7}{3-\frac{4}{n^2}} = \frac{7}{3}$.