1.2 Series (8.2)

Any rational number can be written as a finite sum of fractions: $0.123 = \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3}$.
A real number can also be written as sum of fractions, but when the number is irrational the sum will be infinite: $\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + + +$

Def: A sum of infinite sequence $\{a_n\}_{n=1}^{\infty}$, $S = \sum_{j=1}^{\infty} a_j$ is called **(infinite) series.**

Def: In order to find a sum of infinite sequence $S = \sum_{j=1}^{\infty} a_j$ one defines a sequence of **partial sums** as $\{S_n\}_{n=1}^{\infty} = \left\{ \sum_{j=1}^{n} a_j \right\}_{n=1}^{\infty}$. If the sequence is convergent then $S = \lim_{n\to\infty} S_n$ and the series is called **convergent**. Otherwise the series is **divergent**.

Ex 1. $\sum_{k=1}^{\infty} k = \lim_{n\to\infty} \sum_{k=1}^{n} k = \lim_{n\to\infty} \frac{n(n-1)}{2} = \infty$

Ex 2. Compute $\sum_{k=1}^{\infty} \frac{1}{2^k}$. The partial sums are given as

$$\{S_n\} = \left\{ \sum_{k=1}^{n} \frac{1}{2^k} \right\} =$$

$$\left\{ \frac{1}{2}, \frac{1}{2} + \frac{1}{4} = \frac{2+1}{4} = \frac{3}{4}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4+2+1}{8} = \frac{7}{8}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{8+4+2+1}{16} = \frac{15}{16}, \ldots, \frac{2^n - 1}{2^n} \right\}$$

thus we compute $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n} = \lim_{n\to\infty} \frac{2^x - 1}{2^x} \underset{L'Hospital}{=} \lim_{x\to\infty} \frac{2^x \ln 2}{2^x \ln 2} = 1 \Rightarrow S = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$.

Ex 3. Find $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k}}$. Note, that $\lim_{n\to\infty} \frac{1}{\sqrt{n^2 + k}} = 0$ so it is convergent.

$$1 \leftarrow \frac{n}{\sqrt{n^2 + n}} = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + n}} \le \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k}} \le \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + 1}} = \frac{n}{\sqrt{n^2 + 1}} \to 1 \Rightarrow \lim \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k}} = 1$$

Ex 4. Geometric series: $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots = \frac{a}{1-r}$ when $|r| < 1$, otherwise is divergent.

Ex 5. Telescopic series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

Thm: If $S=\sum_{n=1}^{\infty} a_n$ is convergent series, then

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\left(S_{n+1}-S_n\right)=\lim_{n\to\infty} S_{n+1}-\lim_{n\to\infty} S_n=S-S=0$$

The converse theorem doesn't true, see example below.

Ex 6. We will see later that a hyper harmonic series (p-series) $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent for $p>1$, e.g. $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$. For p=1, even though $\lim_{n\to\infty}\frac{1}{n}=0$, the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$ is diverges since it's partial sum's sequence $S_k=\sum_{n=1}^{k}\frac{1}{n}$ has a divergent subsequence S_{2^k}:

$$S_2=1+\frac{1}{2}$$

$$S_4=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)>1+\frac{1}{2}+\frac{1}{2}=1+\frac{2}{2}$$

$$S_8=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)=1+\frac{3}{2}$$

$$S_{16}=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)>$$

$$>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right)=1+\frac{4}{2}$$

$$S_{2^n}>1+\frac{n}{2}\to\infty$$

The test for Divergence: If $\lim_{n\to\infty} a_n\neq 0$ or DNE then $S=\sum_{n=1}^{\infty} a_n$ is divergent.

Ex 7. $\sum_{n=1}^{\infty}\frac{2^n}{3n+6}$ is diverges since $\lim_{n\to\infty}\frac{2^n}{3n+6}=\lim_{x\to\infty}\frac{2^x}{3x+6}\underset{L'Hospital}{=}\lim_{x\to\infty}\frac{2^x\ln 2}{3}=\infty$

Thm: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the series $\sum_{n=1}^{\infty} ca_n$ (c is constant) and $\sum_{n=1}^{\infty}\left(a_n\pm b_n\right)$. Furthermore: $\sum_{n=1}^{\infty} ca_n=c\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty}\left(a_n\pm b_n\right)=\sum_{n=1}^{\infty} a_n\pm\sum_{n=1}^{\infty} b_n$.

Ex 8. $\sum_{n=1}^{\infty}\frac{3}{n^2}+\frac{7}{2^n}=3\sum_{n=1}^{\infty}\frac{1}{n^2}+7\sum_{n=1}^{\infty}\frac{1}{2^n}=3\cdot\frac{\pi^2}{6}+7\cdot 1=\frac{\pi^2}{2}+7$

Ex 9. $\sum_{n=0}^{\infty}\frac{1}{\left(\sqrt{2}\right)^n}=1+\frac{1}{\sqrt{2}}+\frac{1}{2}+\frac{1}{2\sqrt{2}}+\frac{1}{2^2}+\frac{1}{2^2\sqrt{2}}+\frac{1}{2^3}+\frac{1}{2^3\sqrt{2}}+...=\sum_{n=0}^{\infty}\frac{1}{2^n}+\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{1}{2^n}$