1 Infinite Sequences and Series

1.1 Sequences (8.1)

Def: A **sequence** (or an **infinite sequence**) is a function $f:\mathbb{N}\to\mathbb{R}$ that often given as $f(n)=a_n$. We will often write sequences as $\{a_n\}_{n=1}^{\infty}=\{a_n\}_{n\in\mathbb{N}}$.

Def: A **subsequence** is a sequence that can be derived from another sequence by deleting some elements without changing the order of the remaining elements.

Ex 1. A constant sequence: $\{a_n\}_{n=1}^{\infty}=\{c\}_{n=1}^{\infty}=c,c,c,c\ldots$

Ex 2. Arithmetic sequence (progression): $\{a_n\}_{n=1}^{\infty}=\{a_0+(n-1)d\}_{n=1}^{\infty}=a_0,a_0+d,a_0+2d\ldots$

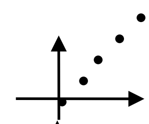
Ex 3. Geometric sequence (progression): $\{a_n\}_{n=1}^{\infty}=\{a_1q^{n-1}\}_{n=1}^{\infty}=a_1,a_1q,a_1q^2\ldots$

Ex 4. Harmonic sequence: $\{a_n\}_{n=1}^{\infty}=\left\{\frac{1}{n}\right\}_{n=1}^{\infty}=1,\frac{1}{2},\frac{1}{3},\frac{1}{4}\ldots$

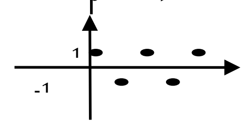
Ex 5. Subsequence of Harmonic sequence: $\{a_{2n}\}_{n=1}^{\infty}=\left\{\frac{1}{2n}\right\}_{n=1}^{\infty}=\frac{1}{2},\frac{1}{4},\frac{1}{6}\ldots$

Graphical representation of sequence:

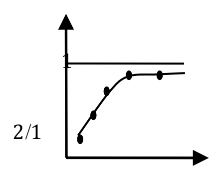
Ex 6. $\{n+1\}=1,2,3,4\ldots$



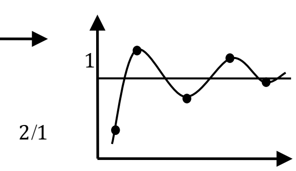
Ex 7. $\left\{(-1)^{n+1}\right\}=1,-1,1,-1\ldots$



Ex 8. $\left\{\frac{n}{n+1}\right\}=\frac{1}{2},\frac{2}{3},\frac{3}{4}\ldots$



Ex 9. $\left\{1+\left(-\frac{1}{2}\right)^n\right\}=\frac{1}{2},1\frac{1}{4},\frac{7}{8},1\frac{1}{16}$



Ex 10. Write the following sequence as $\{a_n\}_{n=1}^{\infty}$

a. $\frac{1}{2},\frac{3}{4},\frac{5}{6},\frac{7}{8}\ldots=\left\{\frac{2n-1}{2n}\right\}_{n=1}^{\infty}$

b. $\frac{1}{3},\frac{1}{9},\frac{1}{27},\frac{1}{81}\ldots=\left\{\frac{1}{3^n}\right\}_{n=1}^{\infty}$

Def: A sequence $\{a_n\}$ has a limit L, i.e. $\lim_{n\to\infty}a_n=L$ or $a_n\to L$ as $n\to\infty$ if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty}a_n$ exists we say the sequence converges (convergent). Otherwise the sequence is diverges (divergent).

Ex 11. $\lim_{n\to\infty}(-1)^{n+1} = \text{DNE}$

Ex 12. $\lim_{n\to\infty}\left(1+\left(-\frac{1}{2}\right)^n\right)=1$

Ex 13.

Thm: If $\lim_{x\to\infty} f(x) = L$, then the sequence $f(n) = a_n$ is convergent and $\lim_{n\to\infty} a_n = L$

Ex 14. $\lim_{n\to\infty} n+1 = \lim_{x\to\infty} x+1 = \infty$

Ex 15. $\lim_{n\to\infty}\frac{n}{n+1} = \lim_{x\to\infty}\frac{x}{x+1} = 1$

Ex 16. $\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{x\to\infty}\frac{\ln x}{x} \underset{L'Hospital\,\infty/\infty}{=} \lim_{x\to\infty}\frac{1/x}{1} = 0$

Limits properties:
If a_n, b_n are convergent and c is constant then

1) $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2) $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$ including $\lim_{n\to\infty} c = c$
3) $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$
4) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \lim_{n\to\infty} b_n \neq 0$
5) $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p, p>0, a_n>0$

The squeeze theorem for sequences: If $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty} b_n = L$

Absolute Value Theorem: $\lim_{n\to\infty}|a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$ (Since $-|a_n| \le a_n \le |a_n|$)

Thm: If *f* is continuous function at L and $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$

Ex 1. $\lim_{n\to\infty}\left(\frac{\pi}{2}+\left(-\frac{1}{2^n}\right)\right) = \frac{\pi}{2} \Rightarrow \lim_{n\to\infty}\sin\left(\frac{\pi}{2}+\left(-\frac{1}{2^n}\right)\right) = \sin\lim_{n\to\infty}\left(\frac{\pi}{2}+\left(-\frac{1}{2^n}\right)\right) = 1$

Ex 2. $\lim_{n\to\infty}\frac{1}{n} = 0 \Rightarrow f(x) = \frac{1}{x} \Rightarrow f\left(\frac{1}{n}\right) = n \Rightarrow \lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{n\to\infty} n = \infty = "f(0)"$

Thm: If a sequence $\{a_n\}_{n=1}^{\infty}$ converges iff subsequences $\{a_{2n}\}_{n=1}^{\infty}$ and $\{a_{2n+1}\}_{n=1}^{\infty}$ does.

Thm: If a sequence $\{a_n\}_{n\in\mathbb{N}}^{\infty}$ converges iff all its subsequences converges.

Corollary: If there exists a divergent subsequence of $\{a_n\}_{n\in\mathbb{N}}^{\infty}$, then $\{a_n\}_{n\in\mathbb{N}}^{\infty}$ diverges.

Thm: $\lim_{n\to\infty} r^n = \begin{cases} 0 & -1<r<1 \\ 1 & r=1 \end{cases}$, When $r>1$ the sequence tends to infinity, and it doesn't exists when $r<-1$ (the last 2 are divergent sequences).

Def: A sequence $\{a_n\}$ is increasing if $a_n \le a_{n+1}$ for all $n \ge 1$. It is called decreasing if it is $a_n \ge a_{n+1}$ for all $n \ge 1$. A sequence is monotonic if it is either increasing or decreasing.

Def: A sequence $\{a_n\}$ is bounded above if there is number M such that $a_n \le M, \forall n \ge 1$. It is bounded below if there is number m such that $a_n \ge m, \forall n \ge 1$. If it is bounded above and below it called bounded sequence.

Thm: Every bounded, monotonic sequence is convergent.

Thm: If $\{b_n\}$ is a subsequence of sequence $\{a_n\}$ obtained by deletion of its first n_0 (finite number) terms. Then $\{a_n\}$ converges iff $\{b_n\}$ does.

Monotonicity tests: 1) $\operatorname{sgn}(a_{n+1}-a_n)$ 2) Does $\frac{a_{n+1}}{a_n}<1$ or $\frac{a_{n+1}}{a_n}>1$?

Ex 3. $n \le n+1 \Rightarrow 0 \le \frac{n}{n+1} \le 1$. Thus $\left\{\frac{n}{n+1}\right\}$ is bounded and therefore convergent.

Ex 4. Check monotonicity of $(3+5n^2)/(n+n^2)$

$$a_{n+1}-a_n = \frac{3+5(n+1)^2}{(n+1)+(n+1)^2} - \frac{3+5n^2}{n+n^2} = \frac{3+5(n^2+2n+1)}{(n+1)+(n^2+2n+1)} - \frac{3+5n^2}{n+n^2} = \frac{5n^2+10n+8}{n^2+3n+2} - \frac{3+5n^2}{n+n^2} =$$

$$= \frac{5n^2+10n+8}{(n+2)(n+1)} - \frac{3+5n^2}{n(n+1)} = \frac{5n^3+10n^2+8n-(3+5n^2)(n+2)}{n(n+2)(n+1)} = \frac{5n^3+10n^2+8n-3n-5n^3-6-10n^2}{n(n+2)(n+1)} =$$

$$= \frac{5n-6}{n(n+2)(n+1)} > 0 \Rightarrow 5n-6>0 \Rightarrow n > \frac{6}{5}$$

The other test:

$$\frac{a_{n+1}}{a_n} = \frac{3+5(n+1)^2}{(n+1)+(n+1)^2}\frac{n+n^2}{3+5n^2} = \frac{3+5n^2+10n+5}{\cancel{(n+1)}(n+2)}\frac{n\cancel{(1+n)}}{3+5n^2} = \frac{5n^3+10n^2+8n}{5n^3+10n^2+3n+6} > 1$$

$$\Leftrightarrow 8n > 3n+6 \Leftrightarrow 5n>6 \Leftrightarrow n > \frac{6}{5}$$

Ex 5. Recursive sequences defined $a_1 = 10$, $a_{n+1} = (2+a_n)/2$

$$a_1 = 10, a_2 = \frac{2+10}{2} = 6, a_3 = \frac{2+6}{2} = 4....$$

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\tfrac{2+a_n}{2} = 1+\tfrac{1}{2}\lim_{n\to\infty} a_n = 1+\tfrac{1}{2}L \Rightarrow L = 1+\tfrac{1}{2}L \Rightarrow \tfrac{1}{2}L = 1 \Rightarrow L = 2$$

Ex 6. $\lim_{n\to\infty}\left(1+\frac{6}{n}\right)^n = e^6$