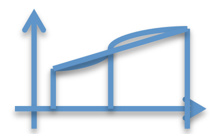
12.9 Approximation of Integration (5.9)

Some functions aren't integrable like $f(x)=\begin{cases}0 & x\in Q\\ 1 & x\notin Q\end{cases}$, the result is depend on choice of x_j^* (see the definition … the limit of sum).

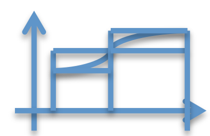
There is functions that are definitely integrable but, the integral cannot be solved in terms of the functions that we are familiar with. Therefore it usually has to be approximated.

We already learned approximations rules (also called quadratures) like the rectangular (end-point) rule and the midpoint rule. We will learn another 2 quadratures a Trapezoidal rule and a Simpson Rule.

12.9.1 The trapezoidal quadrature



The idea of the Trapezoidal rule is to use trapezoidal instead of rectangular approximation. The area of trapeze is given by



$$\frac{a+b}{2}h=\frac{f(x_{i-1})+f(x_i)}{2}\Delta x$$

Thus the **trapezoidal quadrature** is given by

$$\int_a^b f(x)dx=\sum_{j=0}^{n}\int_{a+jh}^{a+(j+1)h} f(x)dx\approx\sum_{j=0}^{n}\Delta x\frac{f(a+j\Delta x)+f(a+(j+1)\Delta x)}{2}=$$

$$=\frac{\Delta x}{2}\left(f(x_0)+2f(x_1)+...2f(x_{n-1})+f(x_n)\right)=\frac{\Delta x}{2}\left(f(x_0)+2\sum_{j=1}^{n-1}f(x_j)+f(x_n)\right)$$

12.9.2 The Simpson Rule:

Consider a polynomial $P_j(x)=A_jx^2+B_jx+C_j$ such that

$$P_j(-\Delta x)=f(x_{2j})=f(x_{2j+1}-\Delta x);\ P_j(0)=f(x_{2j+1});\ P_j(\Delta x)=f(x_{2j+2})=f(x_{2j+1}+\Delta x);$$

Now

$$\int_a^b f(x)dx=\sum_{j=0}^{n-1=2m-1}\int_{x_{2j}}^{x_{2j+2}} f(x)dx=\sum_{j=0}^{n-1}\int_{a+(2j+1)\Delta x-\Delta x}^{a+(2j+1)\Delta x+\Delta x} f(x)dx=\sum_{j=0}^{n-1}\int_{-\Delta x}^{\Delta x} f(x+a+(2j+1)\Delta x)dx$$

$$\approx\sum_{j=0}^{n-1}\int_{-\Delta x}^{\Delta x}A_jx^2+B_jx+C_j\,dx=2\sum_{j=0}^{n-1}\int_0^{\Delta x}A_jx^2+C_j\,dx=2\sum_{j=0}^{n-1}\left(A_j\frac{x^3}{3}+C_jx\right)_0^{\Delta x}=\frac{\Delta x}{3}\sum_{j=0}^{n-1}2A_j\Delta x^2+6C_j$$

Note that

$$\left.\begin{aligned} f\left(x_{2j}\right)&=P\left(-\Delta x\right)=A_j\left(-\Delta x\right)^2+B_j\left(-\Delta x\right)+C_j\\ f\left(x_{2j+1}\right)&=P\left(0\right)=C_j\\ f\left(x_{2j+2}\right)&=P\left(\Delta x\right)=A_j\left(\Delta x\right)^2+B_j\left(\Delta x\right)+C_j\end{aligned}\right\}\Rightarrow f\left(x_{2j}\right)+4f\left(x_{2j+1}\right)+f\left(x_{2j+2}\right)=2A_j\left(\Delta x\right)^2+6C_j$$

We arrive at Simpson Rule:

$$\int_a^b f\left(x\right)dx=\frac{\Delta x}{3}\left(f\left(x_0\right)+4f\left(x_1\right)+f\left(x_2\right)\right)+++\frac{\Delta x}{3}\left(f\left(x_{2(n-1)}\right)+4f\left(x_{2(n-1)+1}\right)+f\left(x_{2(n-1)+2}\right)\right)=$$

$$=\frac{h}{3}\left\{f\left(a\right)+4f\left(x_1\right)+2f\left(x_2\right)+4f\left(x_3\right)+\ldots+f\left(b\right)\right\}=\frac{\Delta x}{3}\left\{f\left(a\right)+2\sum_{j=1}^{n-1}f\left(x_{2j}\right)+4\sum_{j=0}^{n-1}f\left(x_{2j+1}\right)+f\left(b\right)\right\}$$

The Simpson rule can be derived as a combination of a Midpoint Rule M, and a Trapezoid Rule T as following:

$$\int_a^b f\left(x\right)dx=\sum_{j=0}^{n}\int_{a+jh}^{a+(j+1)h}f\left(x\right)dx=\frac{2M+T}{3}\approx\frac{\Delta x}{3}\sum_{j=0}^{n}2f\left(\frac{a+j\Delta x+a+(j+1)\Delta x}{2}\right)+\frac{f\left(a+j\Delta x\right)+f\left(a+(j+1)\Delta x\right)}{2}$$

Ex 12. Approximate $\int_0^2 5x^4\,dx=32$ using Simpson rule with n=4: We have h=2/4, so $x_0=0,\ x_1=0.5,\ x_2=1, x_3=1.5,\ x_4=2$, thus

$$S=\frac{0.5}{3}\left(0+4\cdot5\cdot0.5^4+2\cdot5\quad+\quad4\cdot5\cdot1.5^4+5\cdot2^4\right)=\frac{1}{6}\left(\frac{5}{4}+10+\frac{5\cdot81}{4}+80\right)=32+\frac{1}{12}$$

Error = $\frac{\frac{1}{12}}{32}=\frac{1}{384}\approx2.6\cdot10^{-3}$

Ex 13. Let $f\left(x\right)=\begin{cases}x & 1\le x<2\\ x^2 & 2\le x\le3\end{cases}$, compute $\int_1^3 f\left(x\right)dx$ using n=5 with trapezoidal and Simpson rule. We got $x_i\in\left\{1,1.5,2,2.5,3\right\}$ and $f\left(x_i\right)\in\left\{1,1.5,4,\frac{25}{4},9\right\}$. Thus:

T: $\int_1^3 f\left(x\right)dx=\frac{2}{4}\left(\frac{1}{2}+\frac{3}{2}+4+\frac{25}{4}+\frac{9}{2}\right)=\frac{67}{8}\approx8.375$

S: $\int_1^3 f\left(x\right)dx=\frac{1/2}{3}\left\{1+4\cdot1.5+2\cdot4+4\cdot\frac{25}{4}+9\right\}=\frac{1}{6}\left\{1+6+8+25+9\right\}=\frac{49}{6}\approx8.167$

Analytical: $\int_1^3 f(x)dx = \int_1^2 xdx + \int_2^3 x^2dx = \left.\frac{x^2}{2}\right|_1^2 + \left.\frac{x^3}{3}\right|_2^3 = \frac{4-1}{2} + \frac{9-4}{3} = \frac{47}{6} = 7.8333$

Relative error by method, S: $\frac{8.167-7.833}{7.833} \approx 0.043$, T: $\frac{8.375-7.833}{7.833} \approx 0.069$.

12.9.3 Error Bound of numerical approximation

We won't develop the error bound at this course, just learn the formulas as a fact:

Consider that the approximated integrand has bounded derivatives. For a rectangular rule we need *f'* bounded, say by K_R, for midpoint and trapezoidal rules *f''* by K_M, K_T, and $f^{(4)} \le K_S$ for Simpson. Thus

$$E_R = \frac{(b-a)^2}{n} K_R; \quad E_M = K_M \frac{(b-a)^3}{24n^2}; \quad E_T = K_T \frac{(b-a)^3}{12n^2}; \quad E_S = K_S \frac{(b-a)^5}{180n^4}$$

Ex 14. Find n for which approximation of $\int_1^2 \ln x dx$ will get 10^{-16} error. Do it for Trapezoidal and Simpson rules.

Trapezoidal: $K_T = \max_{x\in[1,2]} \left|\ln'' x\right| = 1$, thus $\left|\frac{K_T}{12}\frac{(b-a)^3}{n}\right| \le \frac{1}{12}\frac{1}{n^2} \le 10^{-6} \Rightarrow n \ge \sqrt{\frac{10^6}{12}} \approx 288.7 \Rightarrow n \ge 289$

Simpson: $\max_{x\in[1,2]} \left|\ln^{(4)} x\right| = 6$, thus $\left|\frac{K_S}{180}\frac{(b-a)^5}{n^4}\right| \le \frac{6}{180}\frac{1}{n^4} = \frac{1}{30}\frac{1}{n^4} \le 10^{-6} \Rightarrow n \ge \sqrt[4]{\frac{10^6}{30}} \approx 13.5 \Rightarrow n \ge 14$