11 Applications of Differentiations

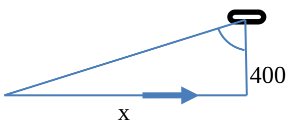
11.1 Related Rates (4.1)

In related rates problem the idea is to compute rate of change of one quantity in terms of rate of change of another quantity. One finds the relationship between the two quantities and then implicitly differentiates it with respect to time using chain rule.

Ex 1. A screen saver displays the outline of a a=3 cm by b=2 cm rectangle and then expands the rectangle in such a way that the 2 cm side is expanding at the rate of 4 cm/sec and the proportions of the rectangle never change. How fast is the area of the rectangle increasing when its dimensions are 12 cm by 8 cm?

Solution: We have $\frac{b}{a}=\frac{2}{3}\Rightarrow a=\frac{3}{2}b$. Thus, the area of the rectangle is given by $S=ab=\frac{3}{2}b^2$. It's given that $\frac{db}{dt}=4$, so $\frac{dS}{dt}=\frac{dS}{db}\frac{db}{dt}=\frac{3}{2}\cdot 2b\cdot\frac{db}{dt}=3b\cdot 4=12b\underset{b=8}{=}96$.

Ex 2. An FBI agent with a powerful spyglass is located in a boat anchored 400 meters offshore. A gangster under surveillance is walking along the shore. Assuming the shoreline is straight and that the gangster is walking at the rate of 2 km/hr, how fast must the FBI agent rotate the spyglass to track the gangster when the gangster is 1 km from the point on the shore nearest to the boat? Convert your answer to degrees/minute.



Solution: We have $\tan\theta=\frac{x}{400}$, differentiating it gives $\frac{1}{\cos^2\theta}\frac{d\theta}{dt}=\frac{1}{400}\frac{dx}{dt}$ and solving for $\frac{d\theta}{dt}$ gives $\frac{d\theta}{dt}=\frac{\cos^2\theta}{400}\frac{dx}{dt}$.

We need $\frac{d\theta}{dt}$ at x=1000, but first we need $\cos^2\theta=\left(\frac{adj}{hyp}\right)^2=\frac{400^2}{400^2+1000^2}=\frac{4^2}{4^2+10^2}=\frac{16}{116}$, so we got $\frac{d\theta}{dt}=\frac{\cos^2\theta}{400}\frac{dx}{dt}=\frac{16/116}{400}\frac{dx}{dt}=\frac{1}{2900}\frac{dx}{dt}$, since $\frac{dx}{dt}=2000\frac{m}{h}$ we arrive at $\frac{d\theta}{dt}=\frac{20}{29}\frac{rad}{hr}=\frac{20}{29}\frac{180/\pi}{60}=\frac{60}{29\pi}\frac{\text{deg}}{\text{min}}\approx 0.6585\frac{\text{deg}}{\text{min}}$.

11.2 Minimum Maximum Values (4.2)

Def: Let $f(x)$ be a function from domain D to range R, i.e. $f: D \to R$ and let $c \in D$. Then $f(c)$ is the

- Absolute maximum of f on D if $f(c) \ge f(x)$ for all $x \in D$
- Absolute minimum of f on D if $f(c) \le f(x)$ for all $x \in D$

Def: Let f be a function from domain D to range R, i.e. $f: D \to R$ and let $c \in \tilde{D} \subset D$. Then $f(c)$ is the

- Local maximum of f on D if $f(c) \ge f(x)$ for all $x \in D$
- Local minimum of f on D if $f(c) \le f(x)$ for all $x \in D$

Ex 3. $\sin x, x^2, x^3$ are functions with many, 1 and no extreme values.

Thm, Extreme Value Theorem: If *f(x)* continuous on closed interval [a,b], the f(x) attain maximum and minimum on [a,b].

Ex 4. Show that $f(x) = \begin{cases} \frac{\sin x}{x} & x \ne 0 \\ 1 & x = 0 \end{cases}$ has a maximum and minimum on [-1,1].

Solution: Since *f(x)* continuous on closed interval [-1,1] by EVT it has a minimum and maximum.

Thm, Fermat's Theorem: If *f(x)* has a local maximum or minimum at *c* and if *f'(c)* exists, then *f'(c)*=0.

Def: Let *f(x)* be defined on D. We say $c \in D$ is a **critical number** of *f(x)* if $f'(c) = 0$ or $f'(c)$ doesn't exists.

Note: If *f(x)* has a local maximum or minimum at c, then *c* is a critical number of *f(x)*.

Alg, How to find absolute maximum & minimum of closed interval:

- Find the values of *f* at the critical numbers.
- Find the values of *f* at end points.

- The larger\smaller value obtained in previous steps is the absolute maximum/minimum.

Ex 5. Let $f(x)=\begin{cases}\dfrac{\sin x}{x} & x\neq 0\\ 1 & x=0\end{cases}$. Find and classify all extreme values of *f(x)* on $[-\pi,\pi]$.

Solution: In order to find extreme values we first find derivatives: on $x\neq 0$ the derivative is $f'(x)=\dfrac{x\cos x-\sin x}{x^2}$, on $x=0$ we need to do little bit more work on use the definition: $f'(0)=\lim\limits_{h\to 0}\dfrac{\frac{\sin h}{h}-1}{h}=\lim\limits_{h\to 0}\dfrac{\frac{\sin h-h}{h}}{h}=\lim\limits_{h\to 0}\dfrac{\sin h-h}{h^2}=0$ (we will see later why it tends to zero).

In order to show that there is no more points where $f'(x)=0$, we first note that zero can come only from the nominator $g(x)=x\cos x-\sin x$ and note that $g'(x)=\cos x-x\sin x-\cos x=-x\sin x\leq 0,\ \forall x\in[-\pi,\pi]$, which mean that $g(x)$ is decreasing on $[-\pi,\pi]$. We now conclude that, since $g(0)=0$ and it decreasing $g(x)\leq 0,\forall x\in[0,\pi]$ and $g(x)\geq 0,\forall x\in[-\pi,0]$. Thus, $f'(x)=0$ only at x=0.

We also need to check the endpoints: $f(\pm\pi)=0$.

Thus, we found that 1 is the absolute maximum and 0 is the absolute minimum.