## 10.8.3 Implicit Differentiation (3.5)

Some function are given in implicit form, for example one defines circle of radius 2 as $x^2+y^2=4$. It is not so difficult to solve this function for $y=\pm\sqrt{4-x^2}$ , but it provides two different functions, which represent the upper and the lower half circle. In additional, some functions are much more difficult to solve for *y* as a function of *x*, for example the **folium of Descartes** $x^3+y^3=6xy$ or something more involved like $\sin y^3 = x^2+\cos xy^2+y$.

When we say that a function f(x) is implicitly defined by equation $x^3+y^3=6xy$ we mean equation $x^3+f(x)^3=6xf(x)$ is true for all values of *x*.

In order to differentiate implicate function we don't need to formulate it explicitly, i.e. we don't need to solve it. Instead, we differentiate both sides of equation.

Ex 2. To find $\frac{dy}{dx}$ of $x^2+y^2=4$ we differentiate both sides to get

$2x+2y(x)y'(x)=0$ and then solve it to get $y'=-\frac{2x}{2y}=-\frac{x}{y}$

Ex 3. To find $\frac{dy}{dx}$ of $x^3+f(x)^3=6xf(x)$ we write $3x^2+3y^2=6(y+xy')$ and solve it for

$$x^2+y^2=2y+2xy'$$
$$y'=\frac{x^2+y^2-2y}{2x}$$

Ex 4. To find $\frac{dy}{dx}$ of $\sin y^3 = x^2+\cos xy^2+y$ we write

$$3y^2y'\cos y^3 = 2x-(y^2+2xyy')\sin xy^2+y'$$
$$3y^2y'\cos y^3 = 2x-y^2\sin xy^2-2xyy'\sin xy^2+y'$$
$$(3y^2\cos y^3+2xy\sin xy^2-1)y' = 2x-y^2\sin xy^2$$
$$y'=\frac{2x-y^2\sin xy^2}{3y^2\cos y^3+2xy\sin xy^2-1}$$

### *10.8.3.1 Another method for implicit differentiation*

There is another way to find derivative of implicit function. One consider implicit function as function of several variables $F(x,y)$, so the equation like $x^2+y^2=25$

becomes $F(x,y)=25$, where $F(x,y)=x^2+y^2$. In this case the derivative become $\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{2x}{2y}=-\frac{x}{y}$, i.e. this provide the same result.

Ex 5. Find dy/dx of $x^2y-y^2x+xy-5=0$

one can define $F(x,y)=x^2y-y^2x+xy-5$ to get $\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{2xy-y^2+y}{x^2-2yx+x}$

the other way is $\begin{array}{l} 2xy+x^2y'-2yy'x-y^2+y'x+y=0 \\ 2xy-y^2+y=-\left(x^2-2yx+x\right)y' \Rightarrow y'=-\frac{2xy-y^2+y}{x^2-2yx+x} \end{array}$

### *10.8.3.2 Derivatives of complicated functions using implicit differentiation*

One can exploit the same idea to find derivatives of complicated functions like

Ex 1. $y=\sqrt{x} \Rightarrow y^2=x \Rightarrow 2yy'=1 \Rightarrow y'=\frac{1}{2y}=\frac{1}{2\sqrt{x}}$

Ex 2. $y=\sqrt[n]{x} \Rightarrow y^n=x \Rightarrow ny^{n-1}y'=1 \Rightarrow y'=\frac{1}{ny^{n-1}}=\frac{1}{n\left(\sqrt[n]{x}\right)^{n-1}}=\frac{1}{n\left(x^{1/n}\right)^{n-1}}=\frac{1}{nx^{1-1/n}}$

## 10.8.4 Derivatives of Inverse Trigonometric Functions (3.6)

One of the more interesting applications of implicit differentiation is differentiation of inverse functions.

$$y=\arcsin x \Rightarrow \sin y=x \Rightarrow y'\cos y=1 \Rightarrow y'=\frac{1}{\cos y}=\frac{1}{\cos\arcsin x}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}$$

$$y=\arccos x \Rightarrow \cos y=x \Rightarrow -y'\sin y=1 \Rightarrow y'=-\frac{1}{\sin y}=-\frac{1}{\sqrt{1-\cos^2 y}}=-\frac{1}{\sqrt{1-x^2}}$$

$$y=\arctan x \Rightarrow \tan y=x \Rightarrow \left(1+\tan^2 y\right)y'=1 \Rightarrow y'=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}$$

Note that all examples hint for the following **Theorem**:

$$y=f^{-1}(x) \Leftrightarrow f(y)=x \Rightarrow f'(y)y'=1 \Rightarrow \frac{d}{dx}f^{-1}(x)=y'=\frac{1}{f'(y)}=\frac{1}{f'\left(f^{-1}(x)\right)}$$

### 10.8.5 Derivatives of Logarithmic Functions

We now find the derivative of logarithmic functions

$$y=\log_a x \Rightarrow a^y = x \Rightarrow a^y(\ln a)y'=1 \Rightarrow y'=\frac{1}{a^y\ln a}=\frac{1}{x\ln a}$$

For a special case $a=e$ we get $\frac{d}{dx}\ln x=\frac{1}{x}$.

Chain rule giveus: $\frac{d}{dx}\ln f(x)=\frac{f'(x)}{f(x)}$

Ex 1. $$\frac{d}{dx}\ln\left(x^2+x-1\right)=\frac{1}{x^2+x-1}\cdot\frac{d}{dx}\left(x^2+x-1\right)=\frac{2x+1}{x^2+x-1}$$

Ex 2. $$\frac{d}{dx}\ln\cos(\sin x)=\frac{1}{\cos(\sin x)}\cdot\left(-\sin(\sin x)\cdot\cos x\right)=-\tan(\sin x)\cos x$$

The following is an interesting result

Ex 3. $$x\neq 0, \frac{d}{dx}\ln|x|=\frac{d}{dx}\begin{cases}\ln x & x>0\\ \ln(-x) & x<0\end{cases}=\begin{cases}\frac{1}{x} & x>0\\ -\frac{1}{x}(-1)=\frac{1}{x} & x<0\end{cases}=\frac{1}{x}$$

### 10.8.6 Logarithmic Differentiation

Another trick to simplify differentiation involving products, quotients and powers:

Ex 4.
$$y=\frac{x^{5/3}\sqrt{x^3+1}}{(5x+3)^3}\Rightarrow \ln y=\ln\frac{x^{5/3}\sqrt{x^3+1}}{(5x+3)^3}=\ln x^{5/3}\sqrt{x^3+1}-\ln(5x+3)^3=$$
$$=\ln x^{5/3}+\ln\sqrt{x^3+1}-3\ln(5x+3)=\frac{5}{3}\ln x+\frac{1}{2}\ln\left(x^3+1\right)-3\ln(5x+3)$$
$$\frac{y'}{y}=\frac{5}{3}\frac{1}{x}+\frac{1}{2}\frac{2x^2}{x^3+1}-3\cdot\frac{5}{5x+3}\Rightarrow y'=\left(\frac{5}{3}\frac{1}{x}+\frac{1}{2}\frac{2x^2}{x^3+1}-3\cdot\frac{5}{5x+3}\right)y=$$
$$=\left(\frac{5}{3}\frac{1}{x}+\frac{1}{2}\frac{2x^2}{x^3+1}-3\cdot\frac{5}{5x+3}\right)\frac{x^{5/3}\sqrt{x^3+1}}{(5x+3)^3}$$

Ex 5. $$y=x^n \Rightarrow |y|=|x|^n \Rightarrow \ln|y|=n\ln|x| \Rightarrow \frac{y'}{y}=n\frac{1}{x}\Rightarrow y'=n\frac{1}{x}y=n\frac{1}{x}x^n=nx^{n-1}$$

Ex 6. $y=x^{\sqrt{x}} \Rightarrow \ln y=\ln x^{\sqrt{x}}=\sqrt{x}\ln x \Rightarrow \frac{y'}{y}=\frac{\ln x}{2\sqrt{x}}+\sqrt{x}\frac{1}{x}=$

$=\frac{\ln x+2}{2\sqrt{x}} \Rightarrow y'=\frac{\ln x+2}{2\sqrt{x}}y=\frac{\ln x+2}{2\sqrt{x}}x^{\sqrt{x}}$

Ex 7. $y=x^{\sin x} \Rightarrow \ln y=\sin x\ln x \Rightarrow \frac{y'}{y}=\frac{\sin x}{x}+\cos x\ln x \Rightarrow y'=\left(\frac{\sin x}{x}+\cos x\ln x\right)x^{\sin x}$

## 10.8.7 The number *e* as a limit

Ex 8. Something more involved

$$\lim_{x\to 0^+} x^x = e^{\ln \lim_{x\to 0^+} x^x} = e^{\lim_{x\to 0^+} \ln x^x} = e^{\lim_{x\to 0^+} x\ln x}$$

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{y\to\infty}\frac{-y}{e^y}=-\lim_{y\to\infty}\frac{y}{e^y}=0 \Leftrightarrow 0\leftarrow\frac{1}{e^y}\le\frac{y}{e^y}\le\frac{y}{y^2}=\frac{1}{y}\to 0$$

$$\Rightarrow \lim_{x\to 0} x^x = e^{\lim_{x\to 0} x\ln x}=e^0=1$$

Consider now the derivative of $f(x)=\ln x$ at x=1:

$$f'(1)=\lim_{x\to 0}\frac{f(1+x)-f(1)}{x}=\lim_{x\to 0}\frac{\ln(1+x)-\ln 1}{x}=\lim_{x\to 0}\frac{1}{x}\ln(1+x)=\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}$$

But we know that $f'(1)=\frac{1}{1}=1$ thus we got $\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=1$.

And we got another and very important limit of

$$e=e^1=e^{\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}}=\lim_{x\to 0}e^{\ln(1+x)^{\frac{1}{x}}}=\lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

Ex 9. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x}=\left(\lim_{y\to 0}(1+y)^{\frac{1}{y}}\right)^2=e^2$

Ex 10. Find $\lim_{x\to\infty}\left(\frac{x^2+2x-1}{x^2-9}\right)^{\left(\frac{3x^2+5x-7}{x+4}\right)}$

$$\frac{3x^2+5x-7}{x+4}=\frac{3x^2+5x-7-20+20}{x+4}=\frac{3\left(x^2-9\right)+5\left(x+4\right)}{x+4}=\frac{3\left(x^2-9\right)}{x+4}+5=\frac{3\left(x^2-9\right)}{x+4}+5=\frac{1}{f\left(x\right)}+5$$

$$\frac{x^2+2x-1}{x^2-9}=\frac{x^2+2x-1-8+8}{x^2-9}=\frac{x^2-9+2x+8}{x^2-9}=1+\frac{2\left(x+4\right)}{x^2-9}=1+6\cdot\frac{x+4}{3\left(x^2-9\right)}=1+6f\left(x\right)$$

$$\lim_{x\to\infty} f\left(x\right)=\lim_{x\to\infty}\frac{x+4}{3\left(x^2-9\right)}=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{4}{x^2}}{3\left(1-\frac{9}{x^2}\right)}=\frac{0}{3}=0\Rightarrow\lim_{x\to\infty}\frac{1}{f\left(x\right)}=\infty$$

$$\Rightarrow\lim_{x\to\infty}\left(\frac{x^2+2x-1}{x^2-9}\right)^{\left(\frac{3x^2+5x-7}{x+4}\right)}=\lim_{f(x)\to0}\left(1+6f\left(x\right)\right)^{\frac{1}{f(x)}+5}=\lim_{y\to0}\left(1+6y\right)^{\frac{1}{y}}\lim_{y\to0}\left(1+6f\left(x\right)\right)^5=$$

$$=\lim_{y\to0}\left(1+6y\right)^{\frac{1}{y}}=\left(\lim_{y\to0}\left(1+6y\right)^{\frac{1}{6y}}\right)^6=e^6$$

Ex 1. Show that $f\left(x\right)=\begin{cases} e^{-1/6} & x=-\frac{3}{2} \\ \left(\frac{4x}{2x-3}\right)^{\frac{1}{2x+3}} & else \end{cases}$ is continuous at $x=-\frac{3}{2}$.

$$\lim_{x\to-3/2}\left(\frac{4x}{2x-3}\right)^{\frac{1}{2x+3}}=\lim_{x\to-3/2}\left(\frac{2x-3+2x+3}{2x-3}\right)^{\frac{1}{2x+3}}=\lim_{x\to-3/2}\left(1+\frac{2x+3}{2x-3}\right)^{\frac{1}{2x+3}}$$

$$\underset{y=2x+3}{=}\lim_{y\to0}\left(1+\frac{y}{y-6}\right)^{\frac{y-6}{y}\frac{1}{y-6}}=\lim_{y\to0}e^{\frac{1}{y-6}}=e^{-1/6}$$