Math1220 Final Exam Summer, 2007 Kelly MacArthur

Name Key Date ____________

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

Each answer is worth 10 points.

Part 1: Answer each of the questions 1 through 8.

1. Find $\frac{dy}{dx}$.

(a) $y=\dfrac{\arctan\left(\dfrac{3}{x^2}\right)}{\ln(5x^2+2)}$

$$\frac{dy}{dx}=\frac{\ln(5x^2+2)\left[\frac{1}{1+\frac{9}{x^4}}\right]\left(\frac{-6}{x^3}\right)-\arctan\left(\frac{3}{x^2}\right)\left(\frac{1}{5x^2+2}\right)(10x)}{\left[\ln(5x^2+2)\right]^2}$$

(b) $y=(\cosh(2x-7)+3x^3-1)^{2+x} = e^{\ln(\cosh(2x-7)+3x^3-1)^{2+x}}$

$$y=e^{\underbrace{(2+x)}_{①}\underbrace{\ln(\cosh(2x-7)+3x^3-1)}_{②}}$$

$$y'=(\cosh(2x-7)+3x^3-1)^{2+x}\Big[1(\ln(\cosh(2x-7)+3x^3-1))$$
$$+(2+x)\left(\frac{1}{\cosh(2x-7)+3x^3-1}\right)(\sinh(2x-7)(2)+9x^2)\Big]$$

$$\frac{dy}{dx}=(\cosh(2x-7)+3x^3-1)^{2+x}\left[\ln(\cosh(2x-7)+3x^3-1)+(2+x)\left(\frac{2\sinh(2x-7)+9x^2}{\cosh(2x-7)+3x^3-1}\right)\right]$$

2. Find the limit, if it exists.

(a) $\lim_{x\to 0^+}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0^+}\left(\frac{x-\sin x}{x\sin x}\right)$

$\left(\begin{array}{c}\infty-\infty\\ \text{indeterminate}\end{array}\right)$

$\overset{L}{=} \lim_{x\to 0^+} \frac{1-\cos x}{x\cos x+\sin x}$

$\overset{L}{=} \lim_{x\to 0^+} \frac{\sin x}{\cos x - x\sin x + \cos x} = \frac{0}{2} = 0$

limit: 0

(b) $\lim_{x\to 0}(x+e^{\frac{x}{2}})^{\frac{2}{x}}$ (1^∞ case - indeterminate)

$y = (x+e^{x/2})^{2/x}$

$\ln y = \frac{2}{x}\ln(x+e^{x/2})$

$\lim_{x\to 0}\frac{2\ln(x+e^{x/2})}{x} \overset{L}{=} \lim_{x\to 0}\frac{\frac{2}{x+e^{x/2}}\left(1+\frac{1}{2}e^{x/2}\right)}{1}$

$\left(\frac{0}{0}\text{ case}\right)$

$= \lim_{x\to 0}\frac{2+e^{x/2}}{x+e^{x/2}} = \frac{2+1}{0+1} = \frac{3}{1} = 3$

$\ln y = 3 \Rightarrow y = e^3$

limit: e^3

3. Determine if each series converges absolutely, converges conditionally, or diverges. Show all your work, state which tests you used, and explain your reasoning.

(a) $\sum_{n=1}^{\infty} \frac{n(-5)^n}{(n+3)!}$

ART

$$\lim_{n\to\infty} \frac{(n+1)5^{n+1}}{(n+4)!} \cdot \frac{(n+3)!}{n(5^n)} = \lim_{n\to\infty} \frac{5(n+1)}{(n+4)n}$$

$$\sim \frac{5n}{n^2} \to 0$$

$\Rightarrow$ absolute convergence

Converges Absolutely or Converges Conditionally or Diverges (circle one)

Why? by ART

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{5n}}{n^3+3}$

LCT $\quad b_n = \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}$ $\quad \sum b_n$ converges because it's a p-series w/ $p = 5/2 > 1$.

$$\lim_{n\to\infty} \frac{\sqrt{5n}}{n^3+3} \cdot \frac{n^{5/2}}{1} = \lim_{n\to\infty} \frac{\sqrt{5}\, n^3}{n^3+3} = \sqrt{5}$$

$0 < \sqrt{5} < \infty$

$\Rightarrow \sum a_n$ also converges

Converges Absolutely or Converges Conditionally or Diverges (circle one)

Why? by LCT + all positive terms

(c) $\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$

Try n^{th} term test

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \overset{L}{=} \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\cancel{\left(\frac{-1}{n^2}\right)}}{\cancel{\left(\frac{-1}{n^2}\right)}} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right)$$

$\left(\frac{0}{0}\text{ case}\right)$

$$= \cos(0) = 1 \neq 0$$

Converges Absolutely or Converges Conditionally or Diverges (circle one)

Why? try n^{th} term test

4. For $f(x)=\dfrac{x^3}{2-x^3}$

(a) Find a power series to represent $f(x)$.

we know $\dfrac{1}{1-\heartsuit} = \sum_{n=0}^{\infty} \heartsuit^n$ $\quad |\heartsuit|<1$

$$f(x) = \frac{x^3}{2}\left(\frac{1}{1-\frac{x^3}{2}}\right)$$

$$= \frac{x^3}{2}\left(\sum_{n=0}^{\infty}\left(\frac{x^3}{2}\right)^n\right) = \frac{x^3}{2}\sum_{n=0}^{\infty}\frac{x^{3n}}{2^n} = \sum_{n=0}^{\infty}\frac{x^{3n+3}}{2^{n+1}}$$

$$\forall \left|\frac{x^3}{2}\right|<1 \iff |x^3|<2$$

$$|x|<\sqrt[3]{2}$$

Power Series: $\sum_{n=0}^{\infty}\dfrac{x^{3n+3}}{2^{n+1}}$

(b) State its convergence set.

check endpts

① $x=\sqrt[3]{2}$ $\quad \sum_{n=0}^{\infty}\dfrac{(2^{1/3})^{3n+3}}{2^{n+1}} = \sum_{n=0}^{\infty}\dfrac{2^{n+1}}{2^{n+1}} = \sum_{n=0}^{\infty} 1 \to \infty$

diverges

② $x=-\sqrt[3]{2}$ $\quad \sum_{n=0}^{\infty}\dfrac{(-1)^{3n+3}(2^{1/3})^{3n+3}}{2^{n+1}} = \sum_{n=0}^{\infty}((-1)^3)^{n+1} = \sum_{n=0}^{\infty}(-1)^{n+1}$

$= 1+-1+1+-1+\dots$

oscillates + diverges

Convergence set: $(-\sqrt[3]{2}, \sqrt[3]{2})$

5. For $f(x)=\ln(3+x)$

(a) Find the Taylor polynomial of order 4 centered about a = 1.

$f(x)=\ln(3+x)$ $\quad f(1)=\ln 4$

$f'(x)=\frac{1}{3+x}$ $\quad f'(1)=\frac{1}{4}$

$f''(x)=\frac{-1}{(3+x)^2}$ $\quad f''(1)=\frac{-1}{16}$

$f'''(x)=\frac{2}{(3+x)^3}$ $\quad f'''(1)=\frac{2}{64}=\frac{1}{32}$

$f^{(4)}(x)=\frac{-6}{(3+x)^4}$ $\quad f^{(4)}(1)=\frac{-6}{4^4}=\frac{-3}{128}$

$f^{(5)}(x)=\frac{24}{(3+x)^5}$

$f(x)=\ln 4+\frac{1}{4}(x-1)+\frac{-1}{16}\left(\frac{1}{2!}\right)(x-1)^2+\frac{1}{32}\left(\frac{1}{3!}\right)(x-1)^3+\frac{-3}{128}\left(\frac{1}{4!}\right)(x-1)^4+\cdots$

Taylor polynomial: $f(x)=\ln 4+\frac{1}{4}(x-1)-\frac{1}{32}(x-1)^2+\frac{1}{192}(x-1)^3-\frac{1}{1024}(x-1)^4+\cdots$

(b) Approximate f(1.5) using the polynomial from (a).

$f(1.5)\simeq \ln 4+\frac{1}{4}\left(\frac{1}{2}\right)+\frac{-1}{32}\left(\frac{1}{2}\right)^2+\frac{1}{192}\left(\frac{1}{2}\right)^3-\frac{1}{1024}\left(\frac{1}{2}\right)^4$

$\cong 1.386294361+0.125-0.0078125+0.00065104167-0.00006103515$

f(1.5) ~= 1.504194

(c) Find a bound for the error in your approximation.

$|R_4(x)|=\left|\frac{f^{(5)}(c)(x-1)^5}{5!}\right|=\left|\frac{\frac{24}{(3+c)^5}(1.5-1)^5}{5!}\right|=\frac{\left(\frac{1}{2}\right)^5}{5(3+c)^5}$

$c\in(1,1.5)$

$=\frac{1}{5(2^5)(3+c)^5}\leq\frac{1}{5(2^5)4^5}\simeq 0.0000061035$

Error bound: 0.0000061035

6. For Cartesian coordinates $(2\sqrt{3},-2)$, find three different ways to represent this point in polar coordinates.

(in Q4)

$x = 2\sqrt{3} \qquad y = -2$

$r^2 = (2\sqrt{3})^2 + (-2)^2 = 12 + 4 = 16 \Rightarrow r = \pm 4$

$\tan\theta = \frac{y}{x} = \frac{-2}{2\sqrt{3}} = \frac{-1}{\sqrt{3}} = \frac{-1/2}{\sqrt{3}/2}$ or $\frac{1/2}{-\sqrt{3}/2}$ $\qquad \frac{5\pi}{6}$ or $\frac{-\pi}{6}$

$(2\sqrt{3},-2) = (4, -\pi/6)$, $(-4, 5\pi/6)$, $(4, 11\pi/6)$

7. Find the slope of the tangent line to the graph of $r = 3\sin(2\theta)$ at $\theta = \frac{\pi}{4}$.

$x = 3\sin(2\theta)\cos\theta \qquad \frac{dx}{d\theta} = -3\sin(2\theta)\sin\theta + 6\cos(2\theta)\cos\theta$

$y = 3\sin(2\theta)\sin\theta \qquad \frac{dy}{d\theta} = 3\sin(2\theta)\cos\theta + 6\cos(2\theta)\sin\theta$

$$m = \frac{dy/d\theta}{dx/d\theta} = \frac{3\sin(2\theta)\cos\theta + 6\cos(2\theta)\sin\theta}{-3\sin(2\theta)\sin\theta + 6\cos(2\theta)\cos\theta}$$

at $\theta = \pi/4$

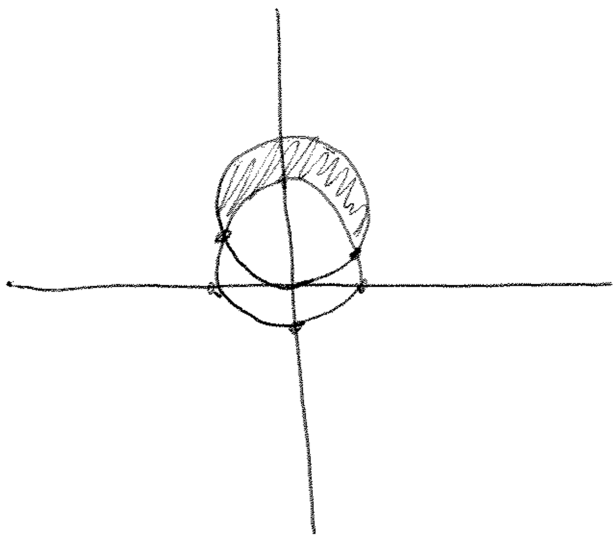
$$m = \frac{3(1)(\sqrt{2}/2) + 6(0)(\frac{\sqrt{2}}{2})}{-3(1)(\frac{\sqrt{2}}{2}) + 6(0)(\sqrt{2}/2)} = \frac{3\sqrt{2}/2}{-3\sqrt{2}/2} = -1$$

Slope = -1

circle limacon

8. For the functions $r=5\sin\theta$ and $r=2+\sin\theta$

(a) Graph the functions on the same coordinate axes and find the points of intersection.



limacon

r	θ
2	0
3	$\pi/2$
2	π
1	$3\pi/2$

intersectn pts

$5\sin\theta = 2+\sin\theta$

$4\sin\theta = 2$

$\sin\theta = 1/2$

$\theta = \pi/6,\ 5\pi/6$

$\Rightarrow r = 5\sin\pi/6$
$= 5(1/2) = 5/2$

Points of intersection: $\left(\frac{5}{2}, \pi/6\right), \left(5/2, 5\pi/6\right)$

(b) Find the area inside $r=5\sin\theta$ and outside $r=2+\sin\theta$.

$$A = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} r_{outer}^2 - r_{inner}^2 \, d\theta\right]$$

$$= \int_{\pi/6}^{\pi/2} (5\sin\theta)^2 - (2+\sin\theta)^2 \, d\theta$$

$$= \int_{\pi/6}^{\pi/2} 25\sin^2\theta - 4 - 4\sin\theta - \sin^2\theta \, d\theta$$

$$= \left(-4\theta + 4\cos\theta\right)\Big|_{\pi/6}^{\pi/2} + \frac{24}{2}\int_{\pi/6}^{\pi/2} 1-\cos 2\theta \, d\theta$$

$$= \left(-4\left(\frac{\pi}{2}\right) + 0\right) - \left(\frac{-4\pi}{6} + 4\left(\frac{\sqrt{3}}{2}\right)\right) + 12\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_{\pi/6}^{\pi/2}$$

$$= \left(-2\pi + \frac{2\pi}{3} - 2\sqrt{3}\right) + 12\left(\frac{\pi}{2} - \left(\frac{\pi}{6} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right)\right) = \frac{-4\pi}{3} - 2\sqrt{3} + 6\pi - 2\pi + 3\sqrt{3}$$

$$= \sqrt{3} + 8\pi/3$$

Area: $\sqrt{3} + 8\pi/3$

Part 2: Choose 5 of the following 7 integrals to evaluate. (Leave your answers in **exact** form, no calculator necessary.) You must choose which problems to grade!

9A. Grade: Yes or No (circle one)

$$\int \frac{x^3}{\sqrt{25-4x^2}}\,dx = \frac{-25}{16}\sqrt{25-4x^2} + \frac{1}{48}\left(25-4x^2\right)^{3/2} + C$$

$$2x = 5\sin\theta$$

$$x = \frac{5}{2}\sin\theta$$

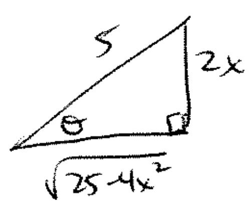
$$dx = \frac{5}{2}\cos\theta\, d\theta$$

$$\sqrt{25-4x^2} = \sqrt{25-25\sin^2\theta}$$

$$= \sqrt{25(1-\sin^2\theta)}$$

$$= \sqrt{25\cos^2\theta}$$

$$= 5\cos\theta$$



$$\sin\theta = \frac{2x}{5}$$

$$\cos\theta = \frac{\sqrt{25-4x^2}}{5}$$

$$\int \frac{\left(\frac{5}{2}\right)^3 \sin^3\theta \left(\frac{5}{2}\right)\cancel{\cos\theta}}{8\cancel{\cos\theta}}\, d\theta$$

$$= \frac{125}{16}\int \sin^3\theta\, d\theta$$

$$= \frac{125}{16}\int \sin\theta\left(1-\cos^2\theta\right) d\theta$$

$$= \frac{125}{16}\left(-\cos\theta\right) - \frac{125}{16}\int \sin\theta\cos^2\theta\, d\theta$$

$$u = \cos\theta$$

$$du = -\sin\theta\, d\theta$$

$$= \frac{-125}{16}\cos\theta + \frac{125}{16}\int u^2\, du$$

$$= \frac{-125}{16}\cos\theta + \frac{125}{48}u^3 + C$$

$$= \frac{-125}{16}\cos\theta + \frac{125}{48}\cos^3\theta + C$$

$$= \frac{-\cancel{125}^{\,25}}{16}\left(\frac{\sqrt{25-4x^2}}{\cancel{5}}\right) + \frac{\cancel{125}}{48}\left(\frac{(25-4x^2)^{3/2}}{\cancel{125}}\right) + C$$

$$= \frac{-25}{16}\sqrt{25-4x^2} + \frac{1}{48}\left(25-4x^2\right)^{3/2} + C$$

9B. Grade: Yes or No (circle one)

$$\int \frac{4x^3+2x^2-3x+6}{x^4+3x^2}\,dx = -\ln|x| - \frac{2}{x} + \frac{5}{2}\ln(x^2+3) + C$$

$$= \int \frac{4x^3+2x^2-3x+6}{x^2(x^2+3)}\,dx = \int \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+3}\,dx$$

$$4x^3+2x^2-3x+6 = Ax(x^2+3) + B(x^2+3) + (Cx+D)x^2$$

$$= Ax^3+3Ax + Bx^2+3B + Cx^3+Dx^2$$

$\Rightarrow$ $4 = A+C$ $\quad$ $2 = B+D$ $\quad$ $-3 = 3A$ $\quad$ $6 = 3B$

$-1 = A$ $\quad$ $2 = B$

$4 = -1 + C$ $\quad$ $2 = 2 + D$

$C = 5$ $\quad$ $D = 0$

$$= \int \frac{-1}{x} + \frac{2}{x^2} + \frac{5x}{x^2+3}\,dx = -\ln|x| - \frac{2}{x} + 5\int \frac{x}{x^2+3}\,dx$$

$u = x^2+3$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= -\ln|x| - \frac{2}{x} + \frac{5}{2}\int \frac{1}{u}\,du$$

$$= -\ln|x| - \frac{2}{x} + \frac{5}{2}\ln(x^2+3) + C$$

$3x-4=0$

$x=4/3$

9C. Grade: Yes or No (circle one)

$1 < 4/3 < 3$

$\int_1^3 \frac{1}{\sqrt[3]{(3x-4)^4}}\,dx =$ diverges

$u = 3x-4$

$\frac{1}{3}\,du = dx$

$$\int \frac{1}{\sqrt[3]{(3x-4)^4}}\,dx = \frac{1}{3}\int u^{-4/3}\,du$$

$$= \frac{1}{3}\left(-3u^{-1/3}\right) + C$$

$$= \frac{-1}{\sqrt[3]{3x-4}} + C$$

$$\int_1^3 \frac{1}{\sqrt[3]{(3x-4)^4}}\,dx = \int_1^{4/3} \frac{1}{\sqrt[3]{(3x-4)^4}}\,dx + \int_{4/3}^3 \frac{1}{\sqrt[3]{(3x-4)^4}}\,dx$$

$$= \lim_{a\to 4/3^-} \int_1^a \frac{1}{\sqrt[3]{(3x-4)^4}}\,dx + \lim_{b\to 4/3^+} \int_b^3 \frac{1}{\sqrt[3]{(3x-4)^4}}\,dx$$

$$= \left.\frac{-1}{\sqrt[3]{3x-4}}\right|_1^a + \left.\frac{-1}{\sqrt[3]{3x-4}}\right|_b^3$$

$$= \lim_{a\to 4/3^-} \frac{-1}{\sqrt[3]{3a-4}} - \left(\frac{-1}{-1}\right) + \frac{-1}{\sqrt[3]{5}} - \lim_{b\to 4/3^+} \frac{-1}{\sqrt[3]{3b-4}}$$

∞ $\quad$ ∞

diverges

9D. Grade: Yes or No (circle one)

$$\int_1^5 \frac{3x^2}{\sqrt{2x-1}}\,dx = \underline{\quad 762 \quad}$$

$u = 2x-1 \Rightarrow u+1 = 2x$ $\quad$ $x=1,\ u=1$

$du = 2dx$ $\quad$ $x = \frac{1}{2}u + \frac{1}{2}$ $\quad$ $x=5,\ u=9$

$\frac{1}{2}du = dx$

$$\frac{1}{2}\int_1^9 \frac{3(\frac{1}{2})^2(u+1)^2}{\sqrt{u}}\,du = \frac{3}{8}\int_1^9 \frac{u^2+2u+1}{\sqrt{u}}\,du$$

$$= \frac{3}{8}\int_1^9 u^{3/2} + 2u^{1/2} + u^{-1/2}\,du$$

$$= \frac{3}{8}\left(\frac{2}{5}u^{5/2} + 2\left(\frac{2}{3}u^{3/2}\right) + 2u^{1/2}\right)\Big|_1^9$$

$$= \frac{3}{8}\left[\left(\frac{2}{5}(\sqrt{9})^5 + \frac{4}{3}(\sqrt{9})^3 + 2\sqrt{9}\right) - \left(\frac{2}{5} + \frac{4}{3} + 2\right)\right]$$

$$= \frac{3}{8}\left[\frac{486}{5} + 36 + 6 - \frac{2}{5} - \frac{4}{3} - 2\right]$$

$$= \frac{3}{8}\left[\frac{484}{5} - \frac{4}{3} + 40\right]$$

$$= \frac{3}{8}\left[\frac{1432}{15} + 40\right]$$

$$= 762$$

9E. Grade: Yes or No (circle one)

$$\int \frac{5(\ln(3x+1))^4}{6x+2}\,dx = \underline{\frac{1}{6}\left(\ln(3x+1)\right)^5 + C}$$

$$u = \ln(3x+1)$$

$$du = \frac{3}{3x+1}\,dx$$

$$\left(\tfrac{1}{2}\right)\frac{1}{3}\,du = \frac{1}{3x+1}\,dx\left(\tfrac{1}{2}\right)$$

$$\frac{1}{6}\,du = \frac{1}{6x+2}\,dx$$

$$= \frac{5}{6}\int u^4\,du$$

$$= \frac{\cancel{5}}{6}\left(\frac{u^5}{\cancel{5}}\right) + C$$

$$= \frac{1}{6}\left(\ln(3x+1)\right)^5 + C$$

9F. Grade: Yes or No (circle one)

$\int (2x+1)\sec^2 x\,dx$ = $(2x+1)\tan x + 2\ln|\cos x| + C$

$u = 2x+1$

$du = 2\,dx$

$v = \tan x$

$dv = \sec^2 x\,dx$

$= (2x+1)\tan x - 2\int \tan x\,dx$

$= (2x+1)\tan x + 2\ln|\cos x| + C$

$\int \tan x\,dx$

$= \int \frac{\sin x}{\cos x}\,dx$

$w = \cos x$

$-dw = \sin x\,dx$

$= -\int \frac{dw}{w}$

$= -\ln|\cos x| + C$

9G. Grade: Yes or No (circle one)

$$\int_1^{\infty} \frac{x}{1+x^4}\,dx = \pi/8$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$x=1,\ u=1$

$x\to\infty,\ u\to\infty$

$$\frac{1}{2}\int_1^{\infty} \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan u \Big|_1^{\infty}$$

$$= \frac{1}{2}\lim_{a\to\infty} \arctan(a) - \frac{1}{2}\arctan(1)$$

$$= \frac{1}{2}(\pi/2) - \frac{1}{2}(\pi/4) = \frac{\pi}{4} - \frac{\pi}{8}$$

$$= \frac{\pi}{8}$$

Part 3: Choose 4 out of the following 6 problems to do. You must choose which problems to grade!

10. Grade: Yes or No (circle one)

assume $x>0$

Solve the differential equation $\frac{dy}{dx}=\frac{y+x^2e^x}{x}$ given that the solution goes through the point (1, 2e).

$$\frac{dy}{dx}=\frac{y}{x}+xe^x$$

$$\frac{dy}{dx}-\frac{1}{x}y=xe^x$$

$$\frac{1}{x}\left(\frac{dy}{dx}\right)-\frac{1}{x^2}y=xe^x\left(\frac{1}{x}\right)$$

$$\frac{d}{dx}\left(\frac{1}{x}y\right)=e^x$$

$$\int d\left(\frac{1}{x}y\right)=\int e^x\,dx$$

$$\frac{1}{x}y=e^x+C$$

$$y=xe^x+Cx$$

$(1,2e)$

$$2e=e+C$$

$$e=C$$

$$y=xe^x+ex$$

$$P(x)=\frac{-1}{x}$$

$$\int P(x)dx=\int\frac{-1}{x}dx$$

$$=-\ln|x|$$

$$=-\ln x$$

$$=\ln\left(\frac{1}{x}\right)$$

$$e^{\int P(x)dx}=e^{\ln\left(\frac{1}{x}\right)}$$

$$\underbrace{=\frac{1}{x}}_{\text{integrating factor}}$$

y = xe^x+ex

11. Grade: Yes or No (circle one)

For the sequence given by $a_n = \frac{n!}{3^n}$

(a) List the first four terms of the sequence.

n	a_n
1	1/3
2	2/9
3	2/9
4	8/27

$\frac{2!}{3^2} = \frac{2}{9}$

$\frac{3!}{3^3} = \frac{2}{3^2} = \frac{2}{9}$

$\frac{4!}{3^4} = \frac{4 \cdot 2}{3^3} = \frac{8}{27}$

(b) Determine whether $\{a_n\}$ converges or diverges. If it converges, find $\lim_{n \to \infty} a_n$.

$\frac{1}{3}, \frac{2}{9}, \frac{2}{9}, \frac{8}{27}$

claim $a_{n+1} > a_n$

$a_{n+1} = \frac{(n+1)!}{3^{n+1}}$

prove $\frac{(n+1)!}{3^{n+1}} \overset{?}{>} \frac{n!}{3^n}$

$\frac{(n+1)!}{3} \overset{?}{>} n!$

$n+1 \overset{?}{>} 3$

$n \overset{?}{>} 2$

$\Rightarrow$ when $n > 2$, $a_{n+1} > a_n \Rightarrow$ increasing sequence $\Rightarrow$ diverges

Converges or Diverges (circle one)

If it converges, $\lim_{n \to \infty} a_n =$ N/A

12. Grade: Yes or No (circle one)

Find the equation of the tangent line to the graph of $y=(1+x^2)^e$ at x = 1.

$$y' = e(1+x^2)^{e-1}(2x)$$

$$y'(1) = e(2)^{e-1}(2) = e(2^e)$$

$$y(1) = (1+1)^e = 2^e$$

$$(1, 2^e) \qquad m = e2^e$$

$$y - 2^e = e2^e(x-1)$$

$$y - 2^e = e2^e x - e2^e$$

$$y = e2^e x - e2^e + 2^e$$

$$y = 2^e(ex - e + 1)$$

Tangent line: $y = 2^e(ex - e + 1)$

13. Grade: Yes or No (circle one)
Use Euler's method with h = 0.2 to approximate the solution to $y' = -2xy$ over [1, 2] given y(1) = 2.

$$f(x,y) = -2xy$$

$$y_{new} = y_{old} + hf(x_{old}, y_{old})$$

$$= y + 0.2(-2xy)$$

$$= y - 0.4xy$$

$$= y(1-0.4x)$$

x	y
1.0	2.0
1.2	1.2
1.4	0.624
1.6	0.27456
1.8	0.0988416
2.0	0.027675648

$y = 2(1-0.4(1)) = 2(0.6) = 1.2$

$y = 1.2(1-0.4(1.2)) = 0.624$

$y = 0.624(1-0.4(1.4)) = 0.27456$

$y = 0.27456(1-0.4(1.6)) = 0.0988416$

$y = 0.0988416(1-0.4(1.8)) = 0.027675648$

14. Grade: Yes or No (circle one)

When you graduate college, you plan on putting \$10,000 into an account that pays interest compounded continuously. You want it to be worth \$85,000 in 40 years. What interest rate must you get to achieve your goal?

$P = \$10,000$

$A = 85,000$

$t = 40$

$r = ?$

$A = Pe^{rt}$

$85000 = 10000e^{r(40)}$

$8.5 = e^{40r}$

$\ln 8.5 = 40r$

$\frac{\ln 8.5}{40} = r$

$0.0535 = r$

Interest rate: 5.35%

15. Grade: Yes or No (circle one)

Three people, Bob, Mary and Kathy, divide an apple as follows. First, they divide it into fourths, each taking a quarter. Then, they divide the leftover quarter into fourths, each taking a quarter and so on. How much apple does each person get?

each person gets

$$\frac{1}{4} + \frac{1}{4}\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{1}{4}\left(\frac{1}{4}\right)\right) + \frac{1}{4}\left(\frac{1}{4}\left(\frac{1}{4}\left(\frac{1}{4}\right)\right)\right) + \cdots$$

$$= \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \cdots$$

geometric series

$$= \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n - 1$$

$$= \frac{1}{1-\frac{1}{4}} - 1 = \frac{1}{\frac{3}{4}} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

Answer: 1/3 :)