6.1 Antiderivatives & Indefinite Integrals

We are now onto "undoing" derivatives. If I know $\frac{dy}{dx} = 2x$, what is y?

$$y = x^2 \quad \text{or} \quad y = x^2+1 \quad \text{or} \quad y = x^2-5$$

In general, $y = x^2 + c$ where $c \in \mathbb{R}$.

$\boxed{\begin{array}{l} \underline{\text{Thm 1}} \\ \text{If } F \text{ \& } G \text{ are differentiable fns on } (a,b) \\ \text{and } F'(x) = G'(x) \;\; \forall\, x \in (a,b), \text{ then } F(x) = G(x) + k \\ \text{for some constant } k. \end{array}}$

i.e. any antiderivative (a.k.a. indefinite integral) returns a family of solutions

Notation: If $F'(x) = f(x)$, then $\int f(x)\,dx + C$.

$\int \, dx$ is the operator symbol.

Ex 1 $\int (x^2+1)\,dx$ (guess)

6.1 (cont)

Indefinite Integrals of Basic Fns

① $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$ $C \in \mathbb{R}$

② $\int e^x\,dx = e^x + C$

③ $\int \frac{1}{x}\,dx = \ln|x| + C$ $x \neq 0$

Indefinite Integral is a Linear Operator

① $\int k f(x)\,dx = k\int f(x)\,dx$

② $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Ex 2 $\int \frac{2}{x^4}\,dx$

Ex 3 $\int \left(7 + \frac{9}{x}\right) dx$

6.1 (cont)

Ex 4

(a) $\int (x^2-5)(x+2)\,dx$

(b) $\int 3\sqrt[5]{w^4}\,dw$

(c) $\int \frac{1-y^2}{3y}\,dy$

(d) $\int e^x - 2x\,dx$

6.1 (cont)

Ex 5

(a) $\frac{du}{dt} = \frac{4}{t} + \frac{t}{4}$

(b) $\int \frac{(2+y^2)^2}{y}\, dy$

(c) Find antiderivative of $\frac{dy}{dx} = \frac{\sqrt{x^3} - 1}{\sqrt{x^3}} \ni y(9) = 4$.

6.2 Integration by Substitution

$$\boxed{\int f'[g(x)]\,g'(x)\,dx = f[g(x)] + C}$$

"u-Substitution" basically undoes chain rule

Ex 1 $\int x^2 e^{x^3-1}\,dx$

Ex 2 $\int 5(5x-1)^{20}\,dx$

6.2 (cont)

Ex 3

(a) $\int \frac{1}{3x-2}\,dx$

(b) $\int e^{8x+1}\,dx$

(c) $\int \frac{x}{\sqrt{x-5}}\,dx$

6.2 (cont)

Ex 4 (a) $\int e^{-x}(1-e^{-x})^4\,dx$

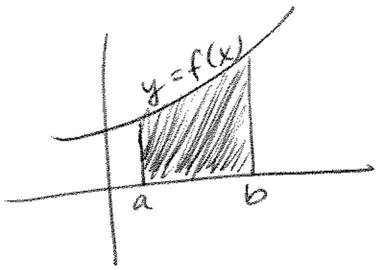
(b) $\int \frac{1}{x\ln x}\,dx$

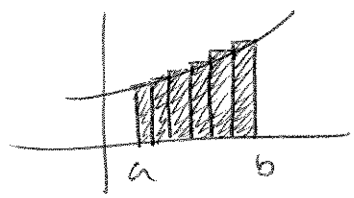
(c) $\int \frac{x^2-1}{(x^3-3x+7)^2}\,dx$

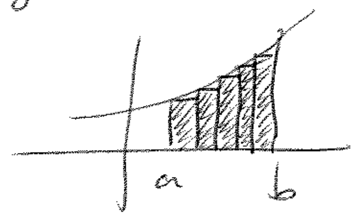
6.4 The Definite Integral

We want to approximate area under a curve.

We can overestimate or underestimate w/ rectangles.







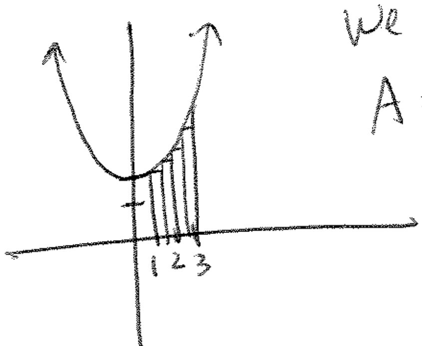
Error in approximating

If $f(x) > 0$ and is either increasing or decreasing on (a, b), then the error in estimating area by rectangles is

$$\varepsilon = \underbrace{|f(b) - f(a)|}_{\Delta y} \underbrace{\left(\frac{b-a}{n}\right)}_{\Delta x}.$$

$\Rightarrow$ as $n \to \infty$, $\varepsilon \to 0$ which means we can get to the exact area by over- or under-estimating area w/ rectangles & then letting the width of our rectangles be infinitessimally small.

EX Consider $y = x^2 + 2$. Find approximate area under curve on $[1, 3]$ by dividing interval into 4 equal subintervals.



We will choose right endpts of intervals.

$$A = f(1.5)\Delta x + f(2)\Delta x + f(2.5)\Delta x + f(3)\Delta x$$

and $\Delta x = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$

6.4 (cont)

$$\Rightarrow A = \left((1.5)^2+2\right)0.5 + \left(2^2+2\right)0.5 + (2.5^2+2)(0.5) + (3^2+2)(0.5)$$

$$= 4.25(0.5) + 6(0.5) + 8.25(0.5) + 11(0.5)$$

$A = 14.75$

(This is called a <u>Riemann sum.</u>)

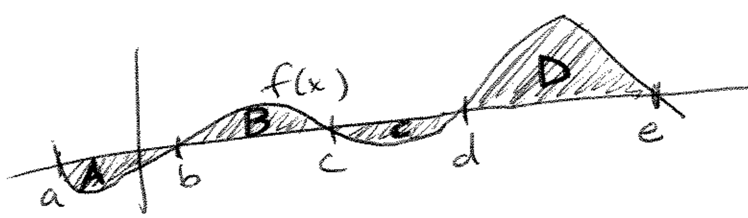
<u>Definite Integral</u>

Let f be a continuous fn on $[a,b]$. The limit I of Riemann sums for f on $[a,b]$ is guaranteed to exist & is called the definite integral of f from a to b. The notation is

$$\int_a^b f(x)\,dx.$$

★ area above x-axis is positive, below x-axis it's negative.

<u>Ex 1</u> Given



$A = 2$
$B = 2.5$
$C = 1.8$
$D = 4$

find

(a) $\int_b^c f(x)\,dx$ (b) $\int_a^d f(x)\,dx$ (c) $\int_c^e f(x)\,dx$

6.4 (cont)

Properties of Definite Integrals

$k \in \mathbb{R}$

① $\int_a^a f(x)\,dx = 0$

② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

③ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

④ $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$

⑤ $\int_a^b f(x) \pm g(x)\,dx$

$= \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

linear operator

Ex 2 Evaluate, given $\int_1^4 x\,dx = 7.5$, $\int_1^4 x^2\,dx = 21$ and $\int_4^5 x^2\,dx = \frac{61}{3}$

(a) $\int_1^4 (7x - 2x^2)\,dx$

(b) $\int_4^1 x(1-x)\,dx$

(c) $\int_5^5 (10 - 4x + 2x^2)\,dx$

6.5 Fundamental Theorem of Calculus

If f is continuous on $[a,b]$ + F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Ex 1 $\int_1^2 3x^2\,dx$

Ex 2 $\int_{-1}^2 (x^2-4x)\,dx$

6.5 (cont)

Ex 3 (a) $\int_0^4 9x^{1/2}\,dx$

(b) $\int_0^1 32x\,(x^2+1)^7\,dx$

(c) $\int_0^2 x e^{x^2}\,dx$

6.5 (cont)

Ex 4 (a) $\displaystyle\int_1^2 \frac{x+1}{2x^2+4x+4}\,dx$

(b) $\displaystyle\int_2^8 \frac{1}{x+1}\,dx$

6.5 (cont)

Defn Average value of a continuous function $f(x)$ on $[a,b]$

$$= \frac{1}{b-a}\int_a^b f(x)\,dx$$

Ex 5 Find avg value over indicated interval.

(a) $f(x) = 4x - 3x^2$ on $[-2, 2]$

(b) $g(x) = 2x+7$ on $[0, 5]$