4.1 e and Continuous Compound Interest

Defn

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{h\to 0}(1+h)^{1/h}$$

$e \simeq 2.718281$ e is irrational

Compound Interest

Let's say you put \$100 into an account earning 10% interest per year + it compounds from year to year.

t = # years
P = \$100 (original invested amount)
A = amount it's worth after t years
r = interest rate (annual)

t	A
0	\$100
1	$100(1+0.10) = 100(1.1)$
2	$[100(1.1)](1+0.1) = 100(1.1)^2$
3	
4	
$\vdots$	
n	

4.1 (cont)

If we have n compoundings per year, the formula becomes

$$\boxed{A = P\left(1+\frac{r}{n}\right)^{nt} \quad \text{Compound Interest}}$$

What if we let $n \to \infty$?

$$A = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}$$

$$= P\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}\right]^{t}$$

Let $h = \frac{r}{n} \Rightarrow n = \frac{r}{h}$

then as $n \to \infty$

$h \to 0$

$$= P\left[\lim_{h\to 0}(1+h)^{r/h}\right]^{t}$$

$$= P\Big[\underbrace{\lim_{h\to 0}(1+h)^{1/h}}_{e}\Big]^{rt}$$

$$\Rightarrow \boxed{A = Pe^{rt} \quad \text{Continuous Compounding}}$$

Ex 1 Calculate $A = 5000e^{0.8t}$ for $t = 1, 4, 10$

4.1 (cont)

Ex 2 Solve for t.

(a) $2=e^{0.03t}$

(b) $3=e^{10t}$

Ex 3 A note will pay \$50,000 at maturity 5 years from now. How much should you be willing to pay for the note now if money is worth 6.4% compounded quarterly?

4.1 (cont)

Ex 4 How long will it take money to double if it is invested at 7% compounded continuously?

Ex 5 Graph $y = e^x$, $y = e^{-x}$ + $y = -e^x$.

4.2 Derivatives of Exponential + logarithmic Fns

$$\boxed{D_x(e^x) = e^x}$$ Derivative of e^x

Note: Big difference between "power" functions of x and "exponential" functions!

Ex 1 Find $f'(x)$.

(a) $f(x) = -5e^x + 3x^2 - 7$

(b) $f(x) = \frac{2}{x^5} - 3e^x + x$

Remember Log Properties

① $y = \log_a b \iff a^y = b$

② $\log_a(bc) = \log_a b + \log_a c$

③ $\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$

④ $\log_a(b^c) = c \log_a b$

$(\ln x = \log_e x)$

4.2 (cont)

Let $f(x) = \ln x$

Then $f'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h}$

$$= \lim_{h\to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{h} = \lim_{h\to 0} \left[\frac{1}{h}\left(\frac{x}{x}\right) \ln\left(1+\frac{h}{x}\right)\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{x}\left(\frac{x}{h}\ln\left(1+\frac{h}{x}\right)\right)\right] = \lim_{h\to 0}\left[\frac{1}{x}\left[\ln\left(1+\frac{h}{x}\right)^{x/h}\right]\right]$$

Let $s = \frac{h}{x}$, then $\frac{1}{s} = \frac{x}{h}$ and as $h \to 0$, $s \to 0$.

$$= \lim_{s\to 0}\left[\frac{1}{x}\left[\ln(1+s)^{1/s}\right]\right] = \frac{1}{x}\ln\Big[\underbrace{\lim_{s\to 0}(1+s)^{1/s}}_{e}\Big]$$

$$= \frac{1}{x}\ln e = \frac{1}{x}(1) = \frac{1}{x}$$

$\Rightarrow \boxed{D_x(\ln x) = \frac{1}{x}}$ Derivative of natural log fn.

Ex 2 Find $f'(x)$.

(a) $f(x) = \ln x^8 - 2x$

(b) $f(x) = \ln x^{10} + 2\ln x$

4.2 (cont)

Change of Base

Let $y = \log_a x$

$\Leftrightarrow \quad a^y = x$

$\ln(a^y) = \ln x$

$y \ln a = \ln x$

$y = \dfrac{\ln x}{\ln a}$

$\Rightarrow \log_a x = \underbrace{\dfrac{1}{\ln a}}_{\text{constant}} (\ln x)$

Derivatives of exponential and logarithmic functions

$D_x(e^x) = e^x$

$D_x(a^x) = (\ln a)\, a^x$

$D_x(\ln x) = \dfrac{1}{x}$

$D_x(\log_a x) = \dfrac{1}{\ln a}\left(\dfrac{1}{x}\right)$

Ex 3 Rewrite f(x) + then find f'(x).

$f(x) = \ln\left(\dfrac{5}{x^4}\right)$

Ex 4 Find $\dfrac{dy}{dx}$, if $y = x^6 - 6^x$

Ex 5 Find the equation of the tangent line to $f(x)$ at indicated value of x.

(a) $f(x) = 1 + \ln x^4$ at $x = e$

(b) $f(x) = 5e^x$ at $x = 1$

4.3 Product + Quotient Rules

Product Rule

If $h(x) = f(x)g(x)$, $f(x)$ + $g(x)$ are differentiable,

then $h'(x) = f'(x)g(x) + f(x)g'(x)$.

Ex 1 Find $\frac{dy}{dx}$.

(a) $y = (2x^2 - 4x + 1)(3x + 5)$

(b) $y = (e^x + 2)(x^2 - 9)$

Quotient Rule

If $h(x) = \frac{f(x)}{g(x)}$, both $f(x)$ + $g(x)$ are differentiable, and $g(x) \neq 0$, then $h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

"low d(hi) − hi d(low) over low squared"

Ex 2 Find y'.

(a) $y = \frac{x^2+3}{2x-1}$

(b) $y = \frac{e^x}{x^3+4x-1}$

4.3 (cont)

Ex 3 Find y'.

(a) $y = \dfrac{10^x}{1+x^{10}}$

(b) $y = \dfrac{2x-1}{(x^3+2)(x^2-3)}$

4.3 (cont)

Ex 4 Find the eqn of the tangent line of $f(x) = (x-2)\ln x$ at $x = 2$.

Ex 5 Find $h'(x)$, given $f(x)$ is unspecified differentiable function.

$$h(x) = \frac{x^2}{f(x)}$$

4.4 The Chain Rule

Remember composite functions

$(f \circ g)(x) = f(g(x))$

e.g. If $f(x) = \sqrt{x}$ and $g(x) = x^2 - 2x + 1$, then

$f(g(x)) =$

$g(f(x)) =$

Chain Rule

If $y = f(u)$ & $u = g(x)$, then

$y = f(g(x))$, and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}, \text{ i.e. } y' = f'[g(x)]\,g'(x)$$

"work from the outside in"

Ex 1 Find y' for $y = (3x^2 - 2x + 5)^6$

4.4 (cont)

Ex 2 Find $\frac{dy}{dx}$.

(a) $y = e^{x^4 + 2x^2 + 5}$

(b) $y = (x - 2\ln x)^4$

(c) $y = \sqrt[3]{3x^2 - 7x}$

Ex 3 Find y'.

(a) $y = [\ln(x^2+3)]^{3/2}$

(b) $y = 2x^2(x^3-3)^4$

(c) $y = \log_5(3x^2 - x^3 + 4x)$

4.5 Implicit Differentiation

Explicit function ⇒ stated directly as "$y =$ stuff"

Implicit function ⇒ stated indirectly; we cannot solve for y by itself.

$y = 3x^2 - 5$ (explicit) $\qquad$ $y^2 - xy + \ln x = 2x^2 - 1$ (implicit)

How do we find $\frac{dy}{dx}$ in implicit functions?

If $y^3 - y^2 = \sqrt{x^2+5}$, then we can take the derivative of both sides of the eqn w.r.t. x. (with respect to)

$$\frac{d}{dx}\left(y^3 - y^2\right) = \frac{d}{dx}\left[\left(x^2+5\right)^{1/2}\right]$$

$$3y^2\frac{dy}{dx} - 2y\frac{dy}{dx} = \frac{1}{2}\left(x^2+5\right)^{-1/2}(2x)$$

$$\frac{dy}{dx}\left(3y^2 - 2y\right) = \frac{x}{\sqrt{x^2+5}}$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+5}\,\left(3y^2-2y\right)}$$

Ex 1 Find y', at the indicated point.

(a) $y^2 - y - 4x = 0$ at $(0,1)$

(b) $2x^3y - x^3 + 5 = 0$ at $(-1,3)$

4.5 (cont)

Ex 2 Find y' at the given point.

(a) $\ln y = 2y^2 - x$ at $(2,1)$

(b) $e^{xy} - 2x = y+1$ at $(0,0)$

4.7 Elasticity of Demand

Defn Relative rate of change of $f(x) = \frac{f'(x)}{f(x)}$

% rate of change $= 100\left(\frac{f'(x)}{f(x)}\right)$

notice $\frac{d}{dx}\left(\ln[f(x)]\right) = \frac{1}{f(x)}\left(f'(x)\right)$ using the chain rule

$\Rightarrow$ relative rate of change can be thought of as the logarithmic derivative of $f(x)$.

Ex 1 If investment A will increase by \$500 in the next year + investment B will increase by \$300 in the same time, which is a better investment? Assume investment A costs \$2000, and investment B costs \$400.

4.7 (cont)

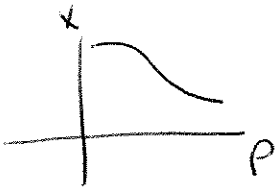
p = price (\$)
x = quantity demanded

Elasticity of Demand

If price and demand are related by $x = f(p)$, then the elasticity of demand is given by

$$E(p) = \frac{-p f'(p)}{f(p)}$$

* both x + p are positive #s
* we typically think of demand decreasing as price increases $\Rightarrow f'(p) < 0$



demand

elasticity of demand is basically the negative ratio of the relative rate of change of demand to the relative rate of change of price.

(since $f'(p) < 0$, then $E(p)$ will be positive)

So

$$E(p) = \frac{\frac{d}{dp}[\ln(f(p))]}{\frac{d}{dp}[\ln p]} = \frac{\frac{1}{f(p)}(f'(p))}{\frac{1}{p}}$$

$$= \frac{p f'(p)}{f(p)}$$

4.7 (cont)

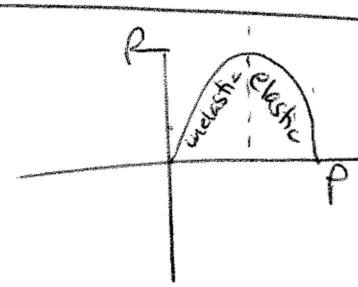
Interpretation

$0 < E(p) < 1$ $R'(p) > 0$	inelastic	change in price produces smaller change in demand (e.g. gasoline)
$E(p) > 1$ $R'(p) < 0$	elastic	change in price produces larger change in demand (e.g. electronics)
$E(p) = 1$	unit	change in price produces same change in demand

R = revenue = (demand)(price)

$= xp$

$\Rightarrow \quad R(p) = f(p)\,p$



Ex 2 For $x = f(p) = 875 - p - 0.05p^2$, is demand elastic, inelastic or unit elastic at the given p-values?

(a) $p = 50$

(b) $p = 100$

4.7 (cont)

Ex 3 Given $x = f(p) = \sqrt{3600 - 2p^2}$ (demand equation), find p values for which demand is elastic and inelastic.

Ex 4 Given $x = f(p) = 10(p-9)^2$ as the demand equation, find the revenue function, graph $R(p)$, and indicate regions of elasticity + inelasticity.