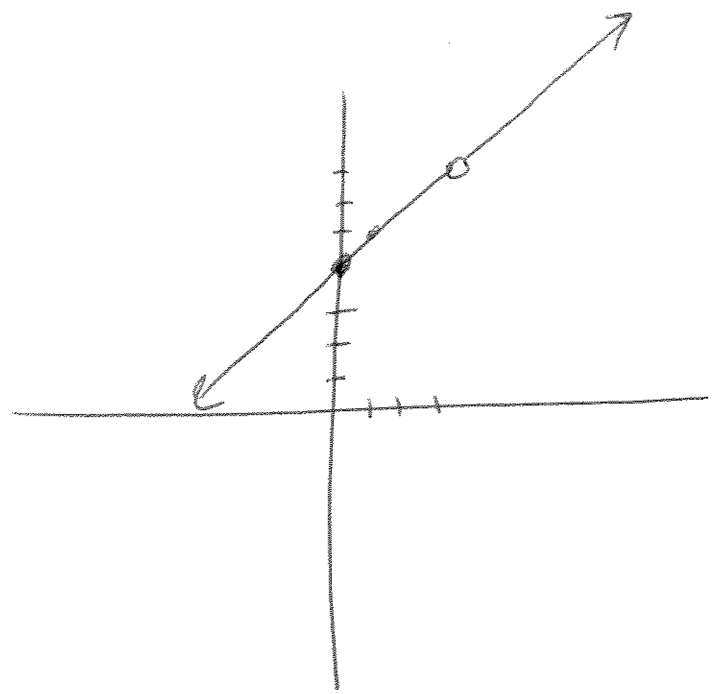
1.1 Introduction to Limits

Consider $f(x) = \dfrac{x^2+x-12}{x-3}$. Note that it's undefined when $x=3$, since $f(3) = \frac{0}{0}$ which isn't a #. What happens as we approach $x=3$?

x	$f(x)$
3.25	7.25
3.2	7.2
3.1	7.1
3.05	7.05
3.01	7.01
3.001	7.001
⋮	
3	?
⋮	
2.99	6.99
2.95	6.95
2.9	6.9
2.8	6.8



So, as x approaches 3, it looks like $f(3) \to 7$.

We'd write $\displaystyle\lim_{x\to 3} \frac{x^2+x-12}{x-3} = 7$. We can compute this algebraically as

$$\lim_{x\to 3} \frac{x^2+x-12}{x-3} =$$

Defn To say $\lim_{x \to c} f(x) = L$ means that when x is near but different from c, then $f(x)$ is near L.

Ex 1 $\lim_{x \to 2} (3x+1)$

Ex 2 $\lim_{x \to 5} \frac{2x^2-7x-15}{x-5}$

Ex 3 $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$

1.1 (continued)

Ex 4 (This is a classic!)

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Argument 1

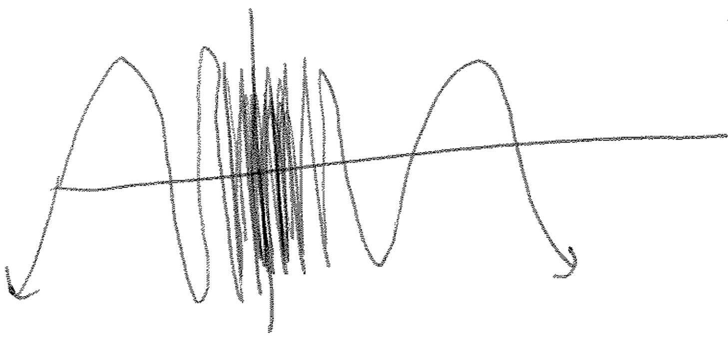
Use your calculator

x	$\frac{\sin x}{x}$
1.0	0.84147
0.5	0.95885
0.1	0.99833
0.01	0.99998
↓	↓
0	?
↑	↑
−0.01	0.99998
−0.1	0.99833
−0.5	0.95885
−1.0	0.84147

Argument 2

1.1 (continued)

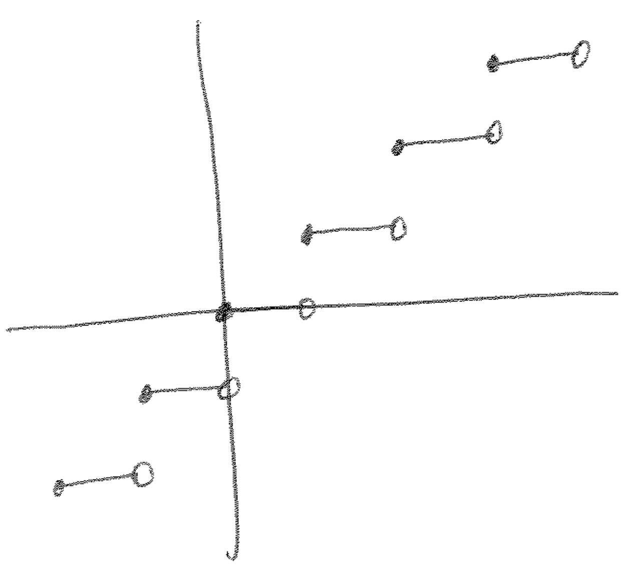
Ex 5 $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$



It wiggles way too much near $x=0$.

Ex 6 $\lim_{x \to 3} [\![x]\!]$

$[\![x]\!]$ = greatest integer function of; it returns the greatest integer $\leq x$.

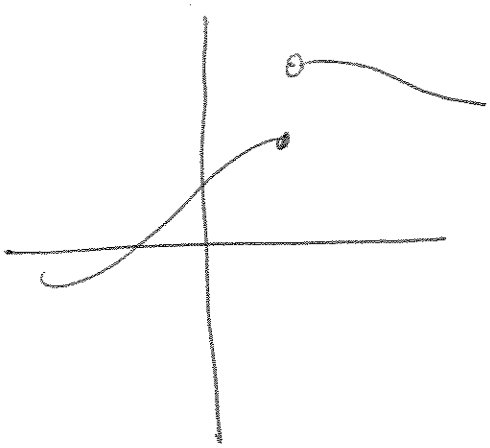


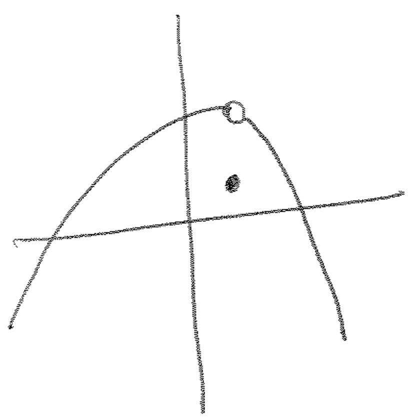
Defn Right + Left Hand Limits

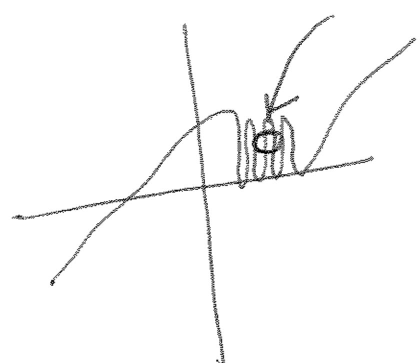
$\lim\limits_{x \to c^+} f(x) = L$ means that when x approaches c from the right side of c, then $f(x)$ is near L.

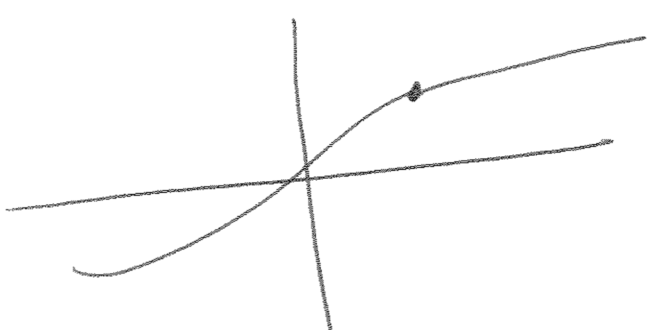
Likewise, $\lim\limits_{x \to c^-} f(x) = L$ means that when x approaches c from the left side of c, then $f(x)$ is near L.

Thm A $L = \lim\limits_{x \to c} f(x)$ iff $\lim\limits_{x \to c^-} f(x) = L = \lim\limits_{x \to c^+} f(x)$









1.3 Limit Theorems

Main Limit Theorem

Assume $n \in \mathbb{Z}^+$, $k \in \mathbb{R}$, and $f(x)$ + $g(x)$ have limits as $x \to c$.

(1) $\lim\limits_{x \to c} k = k$

(2) $\lim\limits_{x \to c} x = c$

(3) $\lim\limits_{x \to c} kf(x) = k \lim\limits_{x \to c} f(x)$

(4) $\lim\limits_{x \to c} [f(x) \pm g(x)] = \lim\limits_{x \to c} f(x) \pm \lim\limits_{x \to c} g(x)$

(5) $\lim\limits_{x \to c} [f(x)g(x)] = \lim\limits_{x \to c} f(x) \lim\limits_{x \to c} g(x)$

(6) $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$, provided $\lim\limits_{x \to c} g(x) \neq 0$

(7) $\lim\limits_{x \to c} [f(x)]^n = [\lim\limits_{x \to c} f(x)]^n$

(8) $\lim\limits_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to c} f(x)}$ if $\lim\limits_{x \to c} f(x) > 0$ $\forall$ n even

Ex 1 $\lim\limits_{x \to 2} (4x^2 - 2x + 1)$

1.3 (continued)

Ex 2 $\lim_{x \to -3} \frac{\sqrt{x^2-1}}{2x}$

Ex 3 (# 22) If $\lim_{x \to a} f(x) = 3$ + $\lim_{x \to a} g(x) = -1$,

find $\lim_{x \to a} \frac{2f(x) - 3g(x)}{f(x) + g(x)}$.

1.3 (continued)

Substitution Thm

If f is a polynomial or rational function, then $\lim_{x \to c} f(x) = f(c)$ assuming $f(c)$ is defined.

Ex 4 $\lim_{x \to -1} \dfrac{3x^4 - 4x^3 + 7x - 5}{2x^2 + 3x + 4}$

Ex 5 $\lim_{x \to 2} \dfrac{3x^3 + 4x + 1}{x^2 - x - 2}$

1.3 (continued)

Squeeze Thm

Let $f, g + h$ be functns satisfying

$f(x) \leq g(x) \leq h(x)$ $\forall x$ near c, except possibly at $x=c$.

If $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$.

~~xxxxx~~

Ex 6 $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$ (Hint: Rationalize numerator.)

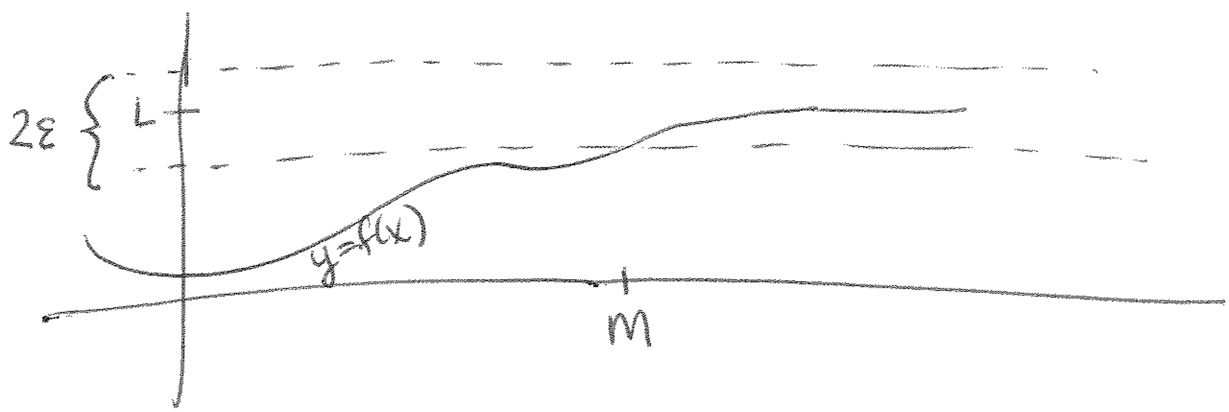
1.5 Limits at Infinity, Infinite Limits

<u>Defn</u> (limit as $x \to \pm\infty$)

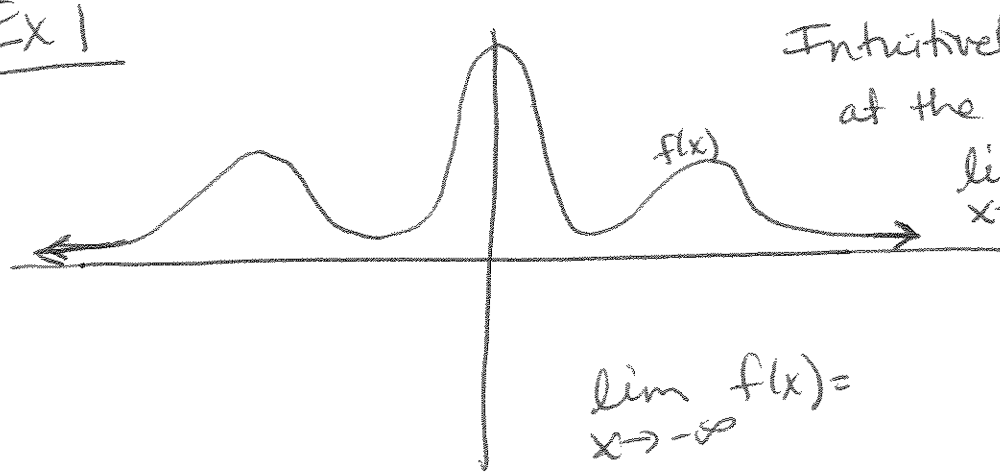
Let f be defined on $[c, \infty)$ (or $(-\infty, c]$) for some $c \in \mathbb{R}$. We say that $\lim_{x \to \infty} f(x) = L$ (or $\lim_{x \to -\infty} f(x) = L$) if $\forall\, \varepsilon > 0$ $\exists$ a corresponding number M $\ni$

$$x > M \text{ (or } x < M) \Rightarrow |f(x) - L| < \varepsilon$$

(M can be dependent on ε.)



<u>Ex 1</u>



Intuitively (looking at the graph)

$\lim_{x \to \infty} f(x) = ?$

$\lim_{x \to -\infty} f(x) =$

Ex 2 Show that if $n \in \mathbb{Z}^+$, then

$|x| > 1$ $\lim_{\cancel{x} \to \infty} \frac{1}{x^n} = 0$. (It's also true that $\lim_{\cancel{x} \to -\infty} \frac{1}{x^n} = 0$)

Let $\varepsilon > 0$ be given. Choose $M = \left(\frac{1}{\varepsilon}\right)^{1/n} = \sqrt[n]{\frac{1}{\varepsilon}}$.

Then $x > M \Rightarrow$ $x^n > M^n \Rightarrow \frac{1}{M^n} > \frac{1}{x^n}$

So $\left|\frac{1}{x^n} - 0\right| = \left|\frac{1}{x^n}\right| < \frac{1}{M^n} = \frac{1}{\left[\left(\frac{1}{\varepsilon}\right)^{1/n}\right]^n} = \frac{1}{1/\varepsilon} = \varepsilon$

i.e. $\left|\frac{1}{x^n} - 0\right| < \varepsilon$. $\Rightarrow$ by defn,

$\lim_{\cancel{x} \to \infty} \frac{1}{x^n} = 0$.

Ex 3 $\lim_{x \to \infty} \frac{2x+3}{x^2+1} = \lim_{x \to \infty} \left(\frac{2x+3}{x^2+1}\right)\left(\frac{1/x^2}{1/x^2}\right)$

1.5 (continued)

Ex 4 $\lim_{x \to \infty} \frac{3x^4 - 2x^3 + 5}{x^3 + 7}$

Ex 5 $\lim_{x \to \infty} \frac{2x^2 + 5x - 1}{x^2 + 3x}$

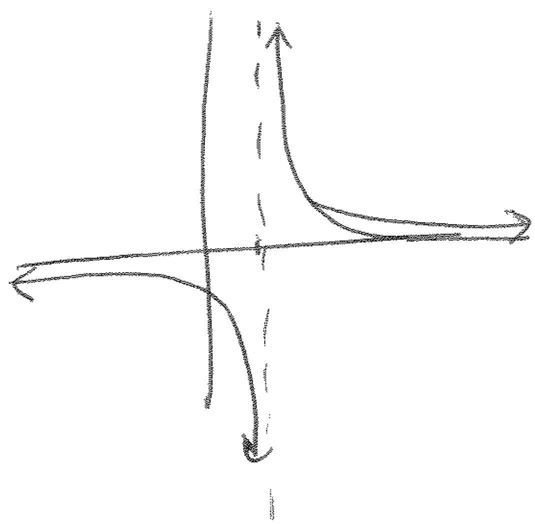
Defn (Infinite limit)

We say $\lim_{x \to c^+} f(x) = \infty$ if $\forall$ positive # M $\exists$ a corresponding $\delta > 0 \ni$

$$0 < x - c < \delta \Rightarrow f(x) > M.$$

i.e. If x is really close to c, $f(x)$ is bigger than some big # M. And, every time we get closer to c, $f(x)$ gets bigger.

1.5 (continued)



$f(x) = \frac{1}{x-1}$ has vertical asymptote $x=1$

You can see from graph,

$\lim_{x\to\infty} f(x) =$

$\lim_{x\to-\infty} f(x) =$

$\lim_{x\to 1^-} f(x) =$

$\lim_{x\to 1^+} f(x) =$

<u>Ex 6</u> Find horizontal & vertical asymptotes of $f(x) = \frac{-5x}{x-2}$

vertical asymptotes $\Rightarrow$ <u>always</u> come from restrictions on x

horizontal asymptotes $\Rightarrow$ describe height (y-value) of graph as x gets outrageously big (in a + or - direction)

vert. asymptote: $x=$

horiz. asymptote: $\lim_{x\to\infty} \frac{-5x}{x-2} =$

1.5 (continued)

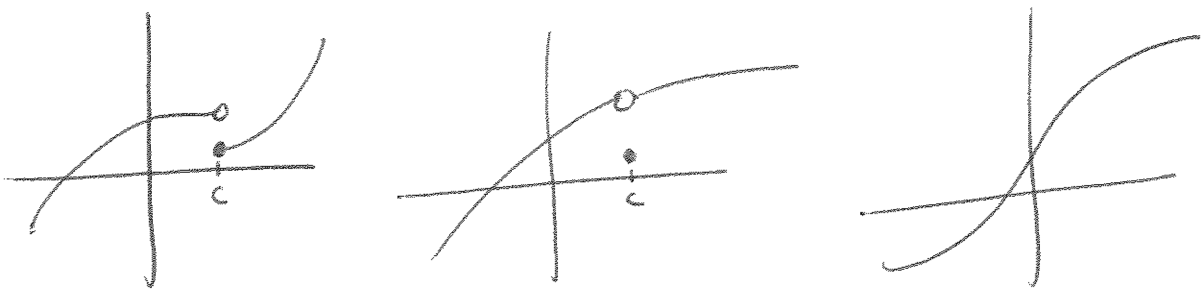
Ex 7 Find ~~as~~ vert. + horiz. asymptotes

for $f(x) = \frac{2x}{\sqrt{x^2+5}}$

Ex 8 (We finally get to use the Squeeze Thm.)

$$\lim_{x \to \infty} x^{-1/2} \sin x$$

1.6 Continuity of Functions



Defn (Continuity at a Point)

Let f be defined on an open interval containing c. We say that f is continuous at c if

$$\lim_{x \to c} f(x) = f(c).$$

i.e. the functn needs to be ① defined at $x=c$, ② the limit needs to exist at $x=c$, and ③ the limit at $x=c$ needs to be exactly the function value.

Continuous Functions

a) All polynomials are continuous everywhere.
b) All rational functns are continuous $\forall x \in$ domain.
c) Absolute value function is continuous $\forall x \in \mathbb{R}$.
d) $f(x) = \sqrt[n]{x}$ continuous $\forall x \in \mathbb{R}$, if n odd.
e) $f(x) = \sqrt[n]{x}$ " $\forall x \in \mathbb{R}$ + non negative if n even.
f) sine + cosine fns are continuous $\forall x \in \mathbb{R}$.
g) $\cot x$, $\csc x$, $\sec x$, and $\tan x$ are continuous $\forall x \in$ domain

1.6 (continued)

More Continuous Fns

If f & g are continuous at c, then so are kf, $f+g$, $f-g$, fg, f/g (if $g(c) \neq 0$), f^n & $\sqrt[n]{f}$ (if $f(c) > 0$ when n even).

Ex 1 State where these fns are continuous.

(a) $g(x) = x^2 - 9$

(b) $h(x) = \sqrt{x-5}$

(c) $f(x) = \dfrac{21-7x}{x-3}$

(d) $p(x) = \begin{cases} -3x+7 & x \leq 3 \\ -2 & x > 3 \end{cases}$

1.6 (continued)

Composite Limit Thm

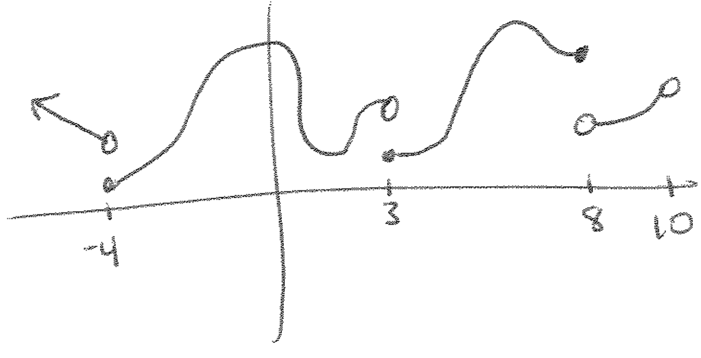
If $\lim_{x\to c} g(x) = L$ & f continuous at L, then
$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L).$$

Ex 2 At what pts are the following functions continuous?

(a) $f(x) = \dfrac{1}{\sqrt{4+x^2}}$

(b) $g(t) = |t-2|$

Ex 3 Where is $f(x)$ continuous? (write answer in interval form)



1.6 (continued)

Ex 4 If $f(x) = \frac{x^2-49}{x-7}$, how do we need to complete the defn for this fn to be continuous everywhere?

Intermediate Value Thm

f is a fn defined on [a,b]. W is # between f(a) + f(b). If f is continuous on [a,b], then ∃ at least one # c between a + b ∋ f(c)=W.