Chapter 2

REAL NUMBERS AND COMPLEX NUMBERS

People have often co-opted words from outside mathematics to describe numbers. Consequently, students of mathematics are usually familiar with nonmathematical meanings of the words *natural, whole, real, complex, imaginary*, and *rational* before they encounter the same words as technical mathematics language. (The word *integer* is a notable exception of not having a common meaning outside mathematics.) Sometimes the mathematical terms are best understood in contrasting pairs. Thus *rational* is contrasted to *irrational, positive* to *negative*, and *real* to *imaginary*. In each case, the first of the pair has a meaning outside mathematics that conveys easier accessibility or greater utility, while the second of the pair evolved from a human tendency to view new things as strange, bad, or unreal.

The excess baggage of knowing nonmathematical meanings can affect how students view these numbers. Probably nowhere is this more pronounced than with the two terms *real* and *imaginary*. To mathematicians, an *imaginary number* is as real as a *real number*, and neither is imaginary in the nonmathematical sense of the word. But students are often influenced by these names to think that real numbers are the actual ones we work with, and to think that imaginary numbers are not numbers at all, but inventions of mathematicians to provide theoretical solutions to equations that have no utility outside mathematics.

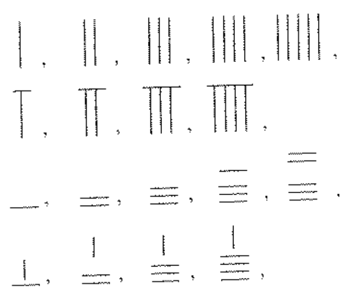
What qualifies a mathematical object to be identified as some type of number? A simple answer is not as easy to obtain as it may first seem. Exactly what basic properties objects called "numbers" should possess can be a subject of debate. Some people would argue that telephone numbers are not numbers (in the mathematical sense) because the operations of addition and multiplication are not meaningful with them. Others would say that these are numbers whose use just happens not to employ arithmetic operations. We address that debate by introducing the idea of a *number system*. A **number system** is a set of objects together with operations (addition and multiplication and perhaps others) and relations (equality and perhaps order) that satisfy some predetermined properties (such as commutativity or the existence of an identity for an operation). With this distinction, we conclude that a telephone number such as 1-800-555-1212 is a number, but it is not a natural number because it does not possess the properties of natural numbers (for example, divisibility or the ability to be added). The Dewey Decimal library classification numbers for books also might be considered as numbers, but they are not rational numbers or real numbers because arithmetic operations are not meaningful with them.

In this chapter, we examine the numbers that make up rational, real, and complex number systems. Our approach is to start from their most familiar geometric representations, the real number line and the complex plane.

Unit 2.1 The Real Numbers

The natural numbers arose historically from the need to count. The extension of the system **N** of natural numbers to the system **Z** of integers was probably prompted by the need to maintain trade accounts. As early as 300 B.C., the Chinese and Indians used rod numerals (Figure 1).

Figure 1



for 1 through 5,

for 6 through 9,

for 10, 20, 30, 40, and 50,

for 60, 70, 80, and 90

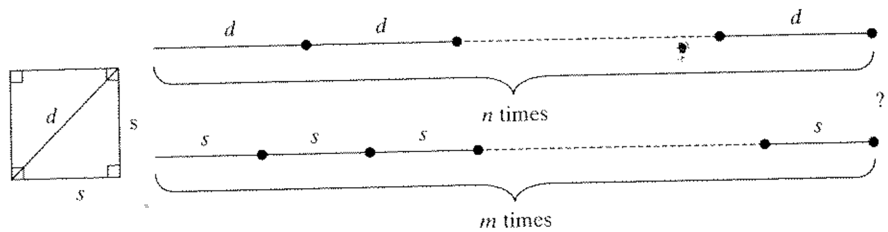
They used right to left positional notation for larger numbers (for example, 6221 was represented by $\perp$ ||=|). They carried out commercial and governmental calculations by using rods of two different colors to distinguish between positive and negative numbers. Interestingly enough, red rods were used for positive numbers, and black rods for negative numbers, the opposite of later uses in Western countries! At first, zero was represented by an empty space in the numeral and later by a more conventional 0. These historical particulars aside, the mathematical importance of the extension from the natural numbers to the integers is that it extends the subtraction $a - b$, which is defined in **N** only for the case in which $a > b$, to arbitrary integers a and b. As a consequence, the equation $x + b = a$ is solvable in **Z** for all choices of a and b in **Z**.

The positive rational numbers were devised to help measure and compare the sizes of objects. The concept of "commensurability" of lengths was fundamental to the early development of geometry by the Greeks. The meaning of commensurability can be explained as follows: Suppose that A and B are two line segments. Then A and B are commensurable if there exist positive integers a and b such that $\frac{\text{length of } A}{\text{length of } B} = \frac{a}{b}$. The ancients used such ratios of positive integers (i.e., what evolved into today's positive rational numbers) not only to compare objects of different lengths but also different weights, areas, and so on.

The measurement of lengths, areas, and volumes of geometric objects became the primary objective of geometric analysis in antiquity. In the sixth century B.C., Pythagoras of Samos and his followers formalized and taught a philosophy of mathematics based on measurement and number ratios that had a deep influence on the evolution of mathematics in Greece for over one hundred years.

The Pythagoreans originally believed that the lengths of all segments in geometric objects are commensurable, and therefore the positive rational numbers are adequate for all measurement purposes. However, the Pythagoreans themselves discovered later that the length s of a side and the length d of a diagonal of a square were not commensurable; that is, they showed that there do not exist natural numbers m and n such that $\frac{d}{s} = \frac{m}{n}$ or, equivalently, $ms = nd$ (see Figure 2).

Figure 2



Question 1: Explain why, in this situation, $\frac{d}{s} = \frac{m}{n}$ is equivalent to $\sqrt{2} = \frac{m}{n}$.

As a result of this unexpected discovery, a major problem developed for the Greek geometers who were rigorously proving theorems. The validity of all theorems involving ratio and proportion was suddenly called into question. This discovery also raised the question of how the lengths of incommensurable segments could be compared. Eudoxus of Cnidus developed a new theory of proportions (360 B.C.), and this theory was incorporated in Euclid's *Elements* (c. 300 B.C.). The contributions of Eudoxus notwithstanding, the subsequent evolution of mathematics in Greece reflected a certain suspicion of measurement and number.

Because $d^2 = 2s^2$ for an isosceles right triangle by the Pythagorean Theorem, the statement that there do not exist natural numbers m and n such that $\frac{d}{s} = \frac{m}{n}$ is equivalent to the statement that there do not exist natural numbers m and n such that $\sqrt{2} = \frac{m}{n}$. In today's language, we say that $\sqrt{2}$ is not a rational number, that is, that $\sqrt{2}$ is an irrational number.

The incommensurability of the lengths of the sides and the diagonal of a square showed that, although the rational numbers arose from the need to measure objects, they are not adequate for that purpose. This inadequacy led very slowly to the use of the broader system of real numbers as the number system for measurement. Although irrational numbers were used in calculations and the solution of algebraic equations, the evolution of the real numbers as an algebraic system did not occur until the nineteenth century. At that time, the evolution of the theory of real functions from its roots in calculus finally necessitated a careful study of the foundations of analysis. This study was independently initiated by Bernhard Bolzano (1781–1848), a priest from Czechoslovakia, and the French mathematician Augustin-Louis Cauchy (pronounced Co'-shee) (1789–1857). The German mathematician Karl Weierstrass (1815–1897) sought to base analysis solidly on a number foundation without appealing to its connections with geometry. But it was the work of Richard Dedekind (1831–1916) and Georg Cantor (1845–1918) of Germany and Charles Méray (1835–1911) of France that finally established precise definitions for the real number system based on the system $\mathbf{Q}$ of rational numbers.

The comparatively slow development of the real numbers as a number system was due in part to the lack of a representation for irrational numbers that was conducive to calculation. The use of decimal representation by Simon Stevin (1548–1620)[1] was a

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[1]Stevin did not invent decimal representation. In fact, a form of decimal representation was used by the ancient Chinese, and mathematical researchers of Stevin's time used it. However, the publication of Stevin's very popular book on decimal representation in 1585 resulted in widespread use of decimal representation among engineers, scientists, and other users of mathematics. The decimal point, to separate the integer and fractional parts of a real number, is first found in the 1619 posthumous publication of John Napier, *Mirifici Logarithmorum Canonis Constructio*, in which he describes how he constructed his tables of logarithms. (In Europe, a comma is still commonly used for this separation.)

great stride forward in this regard. However, another factor that contributed to the slow development of the theory of the real number system was that precise algebraic definitions of *real number* were not given until the years 1869–1879. Méray (1869) and Cantor (1879) constructed real numbers using sequences of rational numbers. Weierstrass (1872) defined real numbers in terms of infinite decimals. In the same year, Dedekind published a treatise in which the real numbers were defined on the basis of partitions of the set **Q** of rational numbers, partitions that are now known as Dedekind cuts. In that treatise, he also developed the algebraic structure of the real number system on the basis of these cuts.

Today there are two basic approaches that can be taken to the theory of real numbers. One is top-down, to assume that there exists a number system that has the properties of a complete ordered field. We take this approach in Chapter 6. The other approach is bottom-up, to construct the real numbers from the rational numbers as done by the mathematicians identified above. We do not carry out all the details of such a construction in this book. But we convey some aspects of the bottom-up approach in this chapter because it helps to illuminate the relationships among (1) the real numbers; (2) decimals, their most common representation; and (3) the geometry of the number line. In Chapter 6 we show how viewing the real numbers as an ordered field relate to (1), (2), and (3).

2.1.1 Rational numbers and irrational numbers

Rational numbers

When we think of rational numbers, we may think of *ratios*, the origin of their name.

Definition A number is **rational** if and only if[2] it can be written as the indicated quotient of two integers.

Numbers written as indicated quotients of two integers, such as $\frac{14}{3}$ or $\frac{-6}{15}$, are rational from the definition. As an immediate consequence of the definition, any integer k is a rational number, because it can be written as $\frac{k}{1}$, the quotient of two integers.

The indicated quotient of a divided by b may be denoted by a slash (a/b), a bar $(\frac{a}{b})$, or a division sign $(a \div b)$. In some countries, a colon $(a:b)$ is used. Either of a/b or $\frac{a}{b}$ is a **fraction**. The bar in $\frac{a}{b}$ also serves as a vinculum, a parenthetical grouping symbol, just as is found in the radical sign $\sqrt{}$. For this reason, $16 + 8/2 + 6$, which has no grouping symbol, equals $16 + 4 + 6$ or 26, while $\frac{16+8}{2+6}$ equals 3. To make the slash act like the fraction bar, parentheses need to be used: $(16 + 8)/(2 + 6) = 3$.

Although every fraction whose numerator and denominator are integers is rational, *a rational number is not the same as a fraction*. The definition of "rational number" does not require that a rational number *must* be written as a quotient of integers, only that it *can* be written that way. Consider the number one-half. It can be written as the fraction $\frac{1}{2}$, or in infinitely many other forms, including notations as diverse as 0.5, $\frac{6}{12}$, $\sin 30°$, $\log \sqrt{10}$, 2^{-1}, $64^{-1/6}$, and $\frac{\pi}{2\pi}$. Of these, only $\frac{6}{12}$ and $\frac{\pi}{2\pi}$ are fractions, and only $\frac{6}{12}$ is an indicated quotient of integers. So rational numbers are not determined by how they look, but by how they *can* look.

[2]Some mathematicians use "if" rather than "if and only if" when it is clear that a word is being defined. For further discussion, see Section 7.1.3.

The various ways of representing $\frac{1}{2}$ give sufficient evidence of the variety of forms in which rational numbers may be written and lead to a natural question: How do we know that -5.4322986, $\sqrt{12}\tan\frac{\pi}{3}$, and $\ln e^{9/8}$ are rational? (They all are.) There is no general procedure, only a requirement. We need to show that each is equal to a quotient of two integers.

Question 2: Write $\frac{\frac{4}{3}}{\frac{7}{2}}$, -5.4322986, $\sqrt{12}\,tan\,\frac{\pi}{3}$, and $\ln e^{9/8}$ as quotients of two integers to show that each is a rational number.

There are infinitely many ways to denote any particular positive rational number even if we insist on writing it as a quotient of two positive integers. For instance, $\frac{4}{3} = \frac{40}{30} = \frac{12}{9} = \frac{124}{93} = \cdots$. But only one of these fractions has the property that its numerator and denominator have no common integer factor greater than 1. This is the fraction that we pick when we say the rational number is **in lowest terms**. Two positive integers are **relatively prime** if they have no common integer factor greater than 1. Thus a fraction in lowest terms has a numerator and denominator that are relatively prime. A negative rational number such as $\frac{-7}{3}$ or $\frac{14}{-6}$ is in lowest terms if the absolute values of the numerator and denominator are relatively prime. Thus either $\frac{-7}{3}$ or $\frac{7}{-3}$ is in lowest terms. Some people prefer positive denominators and would consider $\frac{-7}{3}$ alone to be in lowest terms.

Operations on rational numbers

Why do we care whether a number is rational? One reason is that the algorithms we have for operations with fractions make rational numbers easy to add, subtract, multiply, and divide. For instance, it is almost always easier to multiply by $\sin 30°$ than to multiply by $\sin 40°$, because the first of these equals $\frac{1}{2}$. And, as the following theorem shows, the sum, difference, product, and quotient of two rational numbers is itself a rational number (provided we do not divide by zero). [In the statement of part (b), $\mathbf{Q} - \{0\}$ means the set of rational numbers with 0 removed. In general, if B is a subset of A, then $\mathbf{A}-\mathbf{B}$ is the set of elements in A that are not in B.]

Theorem 2.1

a. The set $\mathbf{Q}$ of rational numbers is closed under addition, subtraction, and multiplication.

b. The set $\mathbf{Q} - \{0\}$ of nonzero rational numbers is closed under division.

Proof:

a. We need show only that the sum, difference, and product of two arbitrary rationals is rational.

Let p and q be the two rational numbers. Since they are rational, there exist integers a, b, c, and d with $p = \frac{a}{b}$ and $q = \frac{c}{d}$, and $b \neq 0$ and $d \neq 0$. Then $p + q = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$. Since the sum and product of two integers is an integer, both $ad + bc$ and bd are integers, and $bd \neq 0$ because neither b nor d is 0. Thus $p + q$ is the quotient of two integers and is rational.

The proofs of the rest of part (a) and part (b) are left to you. ⌟

Estimating rational numbers

In 1978, the following multiple-choice problem was given to a random sample of 13-year-olds participating in the National Assessment of Educational Progress (NAEP).

Problem:	Estimate	$\frac{12}{13} + \frac{7}{8}$.
Choices:	A	1
	B	2
	C	19
	D	21
	E	I don't know

Only 24% selected the correct choice (B). We might conclude that 13-year-olds of that era did not know how to add fractions, but on the Second International Study of Mathematics Achievement in 1981, 84% of 8th graders correctly added two fractions with different denominators in a situation that was not multiple choice. Most researchers have concluded from these examples that the NAEP students tried to answer the above question by adding the fractions blindly without having any idea of the size of the numbers. But then, when they obtained a sum, they didn't know how to connect it with the choices!

Estimating a positive rational number $\frac{a}{b}$ to the nearest integer is easy if it is written as a *mixed number*. A **mixed number** is the sum of an integer and a fraction between 0 and 1, typically written with no space between them. For instance, the mixed number $32\frac{4}{5} = 32 + \frac{4}{5}$. In this case 32 is called the **integer part** of the rational number, and $\frac{4}{5}$ is its **fractional part**.

We can obtain an estimate of the size of a rational number by writing it as a mixed number. For instance, when a car is driven 453 miles between gasoline fill-ups, and it takes 16.8 gallons to fill the tank, the fuel efficiency in miles per gallon is $\frac{453}{16.8}$, a rational number that means little until we determine its integer part. Recall that $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t. When we write a positive rational number t as a mixed fraction, the integer part is $\lfloor t \rfloor$.

To determine $\lfloor t \rfloor$ for a given positive number t, we divide. In the above case, we would divide 16.8 into 453. One way to do this is to multiply both divisor and dividend by 10 to obtain integers, and divide 168 into 4530. To obtain the integer part of the quotient, we may repeatedly subtract 168 from 4530, or we might use long division, or today most people would use a calculator. Regardless of the process, we find that $\lfloor \frac{4530}{168} \rfloor = 26$. This indicates that $\frac{4530}{168} = 26 + \frac{r}{168}$, where r is the remainder that needs to be determined. Multiplying both sides by 168, $4530 = 168 \cdot 26 + r$, from which $r = 162$. So $\frac{4530}{168} = 26\frac{162}{168} = 26\frac{27}{28}$.

The theorem about integers that guarantees a unique quotient and unique remainder in this process is called the *Division Algorithm*. We deduce the Division Algorithm from properties of natural numbers and discuss it in detail in Section 5.2.1 but state it here since we use it repeatedly in this chapter.

Theorem 5.3 **(Division Algorithm):** If a and b are integers and if $b > 0$, then there exist unique integers q and r such that $a = bq + r$ and $0 \leq r < b$.

Dividing both sides by b, we obtain the form of the Division Algorithm that we used above.

Corollary (Alternate Form of Division Algorithm): If a and b are integers and if $b > 0$, then there exist unique integers q and r such that $\frac{a}{b} = q + \frac{r}{b}$, with $0 \leq r < b$.

In some computer languages, q and r are denoted as **a div b** and **a mod b**. Notice in the corollary that since $r < b$, we have $\frac{r}{b} < 1$, and so $q = \lfloor \frac{a}{b} \rfloor$ if $a > 0$. In this way, the Division Algorithm explains why every rational number t is either an integer or between two consecutive integers, shows how those integers can be calculated, and also determines how to write t as a mixed number.

For instance, if $a = -164$ and $b = 5$, then we are looking for q and r with $-164 = 5q + r$ and $0 \le r \le 5$. $q = \lfloor \frac{-164}{5} \rfloor = \lfloor -32\frac{4}{5} \rfloor = -33$, from which $r = 1$. Thus $\frac{-164}{5}$ is between -33 and -32, and $\frac{-164}{5} = -32\frac{4}{5}$. So we say the integer part of $-32\frac{4}{5}$ is -33, and the fractional part is $\frac{1}{5}$. (Some people say the integer part of $-32\frac{4}{5}$ is -32, and the fractional part is $-\frac{4}{5}$.)

In the next section, we show how this process also enables us to write rational numbers as decimals. But now we turn to irrational numbers.

Irrational numbers

An **irrational number** is a real number that is not a rational number.

Today, irrational numbers are found throughout the study of middle school and high school mathematics. They include the common logarithms of all positive integers except the integer powers of 10 and the natural logarithms of all positive integers greater than 1; the sines, cosines, and tangents of all integer degrees except for some divisible by 15°; the square roots and cube roots of most of the positive integers; and, of course, π and all its rational multiples. They are lengths of segments in geometry, coefficients in formulas for area and volume of common figures, and values of some of the most important functions in mathematics. Pick the coefficients a, b, and c of the quadratic equation $ax^2 + bx + c = 0$ at random and you are more likely to have solutions that are not rational than to have rational solutions. Indeed, as we show in Section 2.1.3, there are far more irrational numbers than rational numbers.

The existence of irrational numbers

In this book, we give many proofs of the existence of irrational numbers. We show:

Square roots of positive integers are either positive integers or irrational. (Theorem 2.2)

Numbers represented by infinite nonrepeating decimals are irrational. (Section 2.1.2)

e is irrational. (Section 2.1.3)

The number of irrational numbers is infinite and not countable. (Section 2.1.4)

Roots of many polynomial equations are irrational. (Section 2.1.4)

We also discuss the irrationality of other specific numbers, such as π.

Our first proof shows that the square root of any positive integer that is not a perfect square is an irrational number. It relies on two facts: (1) Except for the order of the factors, every integer has a unique factorization into primes. This statement, known as the Fundamental Theorem of Arithmetic, is proved in Section 5.2.4. (2) If integers a and b are relatively prime (i.e., have no common integer factors >1), then a^2 and b^2 are relatively prime. Fact (2) follows quite quickly from (1).

Theorem 2.2 Let n be a positive integer. Then $\sqrt{n}$ is either an integer or it is irrational.

Proof: Obviously the proof has to take into account the meaning of "square root" and of "irrational". The proof is indirect. Suppose n is a positive integer, $\sqrt{n}$ is not an integer, and $\sqrt{n}$ is rational. Then there exist relatively prime integers a and

b with $\sqrt{n} = \frac{a}{b}$. Squaring both sides (using the definition of "square root"), $n = \frac{a^2}{b^2}$, from which $nb^2 = a^2$. Now if we factor a and b into primes, there are no common factors. So the factorizations of a^2 and b^2 have no common factors. Consequently, the factorizations of the equal numbers nb^2 and a^2 are different. Since two different factorizations of a^2 are impossible by the Fundamental Theorem of Arithmetic, the supposition must be false. So if $\sqrt{n}$ is not an integer, it must be irrational. ┘

As a result of Theorem 2.2, numbers such as $\sqrt{83}$ and $\sqrt{21}$ are irrational.

Theorem 2.2 can be considered to be a theorem about solutions to equations. It is equivalent to asserting that if p is not a perfect square, the polynomial equation $x^2 - p = 0$ has no rational solutions. This, in turn, is a special case of the *Rational Root Theorem* found in many textbooks.[3]

Once a number has been shown to be irrational, it can be used to produce many other irrational numbers.

Theorem 2.3 Let s be any nonzero rational number and v be any irrational number. Then $s + v$, $s - v$, sv, and $\frac{s}{v}$ are irrational.

Proof: The proof is indirect. Suppose s and v are as given and $s + v$ is rational. Then $(s + v) - s$ is rational from Theorem 2.1(a). But $(s + v) - s = v$, which is irrational. This contradiction shows that $s + v$ must be irrational. The other parts of the theorem can be proved in a similar fashion. ┘

Theorems 2.2 and 2.3 together show that such numbers as $2 + \sqrt{3}$ and $\dfrac{-14}{2 + \frac{3}{5}\sqrt{7}}$ are irrational.

Operations on irrational numbers

On occasion, irrational numbers are nicely related. For instance, the product of two square roots of positive integers is another square root of a positive integer, as can be seen from the identity $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$. But this does not work for sums of square roots. Even as simple a sum as $\sqrt{2} + \sqrt{3}$ does not equal the square root of an integer. Relationships among irrationals are always tied to their origins. For instance, an identity such as $\log(xy) = \log x + \log y$ may involve irrational numbers, but its truth is traceable back to properties of logarithms, not to properties of irrational numbers.

Indeed, sums, differences, products, and quotients of irrational numbers may be rational or irrational. So the set $\mathbf{I}$ of irrational numbers is not closed under any of the operations of arithmetic.

[3] The Rational Root Theorem: If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ is a polynomial with integer coefficients, and $\frac{a}{b}$ is a rational solution to $p(x) = 0$ in lowest terms, then a is a factor of a_0, and b is a factor of a_n.

2.1.1 Problems

1. Write each number as the quotient of two integers. If the number can be written as a mixed number, identify its integer part and fractional part in lowest terms.

a. 3.14159 b. $\frac{35789.22}{47.6}$

c. $\log_{10} \sqrt[3]{100}$ d. $-6\frac{3}{4}$

2. Give examples to show that the following statements are not true for all positive rational numbers x and y.

a. $\lfloor x + \frac{1}{10} \rfloor = \lfloor x \rfloor$

b. $\lfloor x \rfloor + \lfloor y \rfloor = \lfloor x + y \rfloor$

c. $\lfloor x \rfloor - \lfloor y \rfloor = \lfloor x - y \rfloor$

3. Expressions may look irrational yet still be rational.

a. Write $\sqrt{3 + \sqrt{7} - \sqrt{8 - 2\sqrt{7}}}$ as the quotient of two integers.

b. Make up another example of the same type as part **a**.

c. Generalize parts **a** and **b**.

4. a. Prove Theorem 2.1(a) for subtraction.

b. Prove Theorem 2.1(a) for multiplication.

c. Prove Theorem 2.1(b).

5. a. Is the set of rational numbers closed under the operation of exponentiation? That is, for all rational numbers a and b, is a^b rational? Why or why not?

b. Is the set of irrational numbers closed under the operation of exponentiation?

6. Can you use the process in the proof of Theorem 2.2 to prove that the square root of 25 is irrational? If so, show how. If not, indicate where the proof breaks down.

7. Modify the proof of Theorem 2.2 to show that the positive nth root of any prime is irrational.

8. Prove or disprove the claim of a student that the sides of any right triangle can be written in the form $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{a + b}$.

9. Let s be a nonzero rational number and v be irrational.

a. Prove that $s - v$ is irrational.

b. Prove that sv is irrational.

c. Prove that $\frac{s}{v}$ is irrational.

10. a. Give an example of two different irrational numbers whose sum is a rational number.

b. Give an example of two different irrational numbers v_1 and v_2 such that $\frac{v_1}{v_2}$ is rational.

11. a. Prove that if a, b, and c are integers and $\sqrt{b^2 - 4ac}$ is not an integer, then $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is irrational.

b. Give a specific example of a quadratic equation whose solutions are proved irrational by applying the result of part **a**.

12. Although $\sqrt{2} + \sqrt{3}$ does not equal the square root of an integer, $\sqrt{27} + \sqrt{48}$ does.

a. What integer's square root equals $\sqrt{27} + \sqrt{48}$, and why?

b. Make up another example like $\sqrt{27} + \sqrt{48}$.

c. Find every set of different positive integers p, q, and r all less than 100 such that p and q are not perfect squares and $\sqrt{p} + \sqrt{q} = \sqrt{r}$.

13. a. Consider $\sqrt{n}$, where n is an integer and $1 \leq n \leq 10$. How many of these ten numbers are irrational?

b. An integer n is randomly chosen from 1 to k^2, where k is an integer. What is the probability that n is a perfect square? What is the probability that $\sqrt{n}$ is irrational?

ANSWERS TO QUESTIONS

1. Suppose m and n are natural numbers. Since $d^2 = 2s^2$, $\frac{d^2}{s^2} = 2$. Since d and s are (positive) lengths, $\sqrt{\frac{d^2}{s^2}} = \sqrt{2}$, from which $\frac{d}{s} = \sqrt{2}$. So if $\frac{d}{s} \neq \frac{m}{n}$, then $\sqrt{2} \neq \frac{m}{n}$. **2.** $\frac{8}{21}$, $\frac{-54322986}{10000000}$, $\frac{6}{1}$, $\frac{9}{8}$

2.1.2 The number line and decimal representation of real numbers

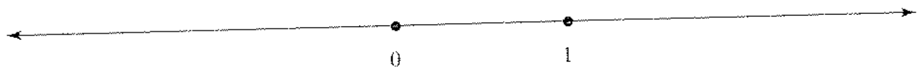
In school algebra, real numbers are commonly described as numbers that can be represented by finite or infinite decimals. In geometry, they are introduced as numbers that are in one-to-one correspondence with the points on a line. In higher mathematics, real numbers may be defined in terms of rational numbers by least upper bounds, sums of infinite series, nested intervals, or Dedekind cuts. In this section, we connect these more advanced ideas to decimals and the number line. Our approach is to begin with the number line, describe rational numbers as series of the form $\sum_{i=-\infty}^{n} a_i b^i$ with $b = 10$, and then use the Nested Interval Property of the number line to obtain real numbers as decimals.

The number line

Although we often think of the arithmetic/algebra aspect of mathematics as being separate from the geometric aspect of mathematics, there is a fundamental interplay between the two. Arithmetic and algebra provide models for geometry, and vice versa. Geometry can be modeled algebraically through coordinates or vectors or complex numbers, and these algebraic models can lead to new geometric insights. In turn, geometric representations of real and complex numbers play important roles in our understanding of these numbers.

The *number line* or *real line* is a geometric model or representation of the system $\mathbf{R}$ of real numbers. In this model, we begin with a straight line and select two points on that line that represent the integers 0 and 1, as in Figure 3.

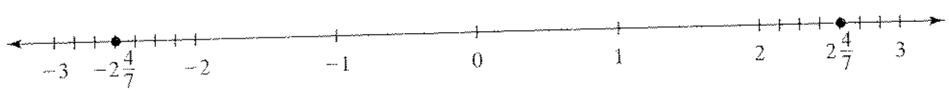
Figure 3



Then we represent the successive positive integers $2, 3, 4, \ldots$ by equally spaced points to the right of 1, and the successive negative integers by equally spaced points to the left of the point 0.

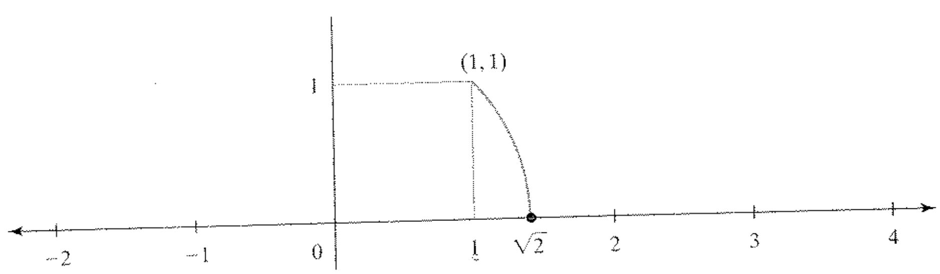
A positive rational number x that is not an integer can be represented by thinking of it as a mixed number. Suppose $x = \frac{a}{b}$, where a and b are integers; then $x = \frac{a}{b} = q + \frac{r}{b}$, where $q = \lfloor x \rfloor$ and $0 < r < b$. On the number line, x is represented by the point that divides the segment from q to $q + 1$ in the ratio $\frac{r}{b}$. For example, the rational number $\frac{18}{7} = 2 + \frac{4}{7}$. So $\frac{18}{7}$ is represented by the point between 2 and 3 on the number line that divides the segment from 2 to 3 in the ratio $\frac{4}{7}$. A negative rational number x is represented by the point on the other side of 0 at the same distance from 0 as the point representing $|x|$ (Figure 4).

Figure 4



This number line model also allows us to represent irrational numbers by points on the number line. For example, we know that $\sqrt{2}$ is the length of either diagonal of a square with side length equal to 1. Consequently, the point on the real line corresponding to $\sqrt{2}$ can be constructed as the intersection of the real line and the arc centered at $(0, 0)$ and containing $(1, 1)$, as indicated in Figure 5.

Figure 5



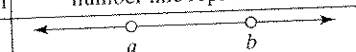
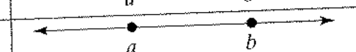
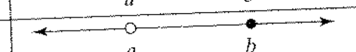
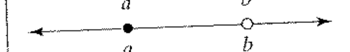
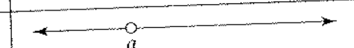
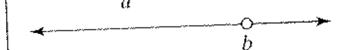
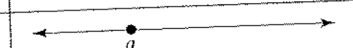
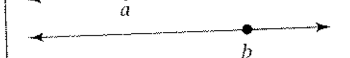
In this way, if an irrational number can be identified with a length, we can find the point on the number line corresponding to it.

This geometric description of the system $\mathbf{R}$ of real numbers is quite adequate for many purposes and is the basis of much of our intuition about real numbers. Most of us find this model to be natural and intuitive because of our experience with concrete scales and measuring devices such as thermometers and rulers. In this section and the next, we use this model of the real number system $x \in \mathbf{R}$ to describe its features. In Section 2.1.4, we outline how the system of real numbers can be defined without reference to the number line model and indicate why it was necessary to develop such definitions.

Intervals

An **interval** of numbers is a set containing all numbers between two given numbers together with one, both, or neither of the given numbers. Therefore, an interval can be modeled on the number line by a segment or ray with or without its endpoints. An interval is **closed** if it contains its endpoints and **open** if it does not contain them. The **length** of an interval with endpoints a and b and with $a < b$ is $b - a$. Interval notation and terminology are summarized in Table 1. (Read "$\in$" as "is an element of", $+\infty$ as "positive infinity", and $-\infty$ as "minus infinity".)

Table 1 Intervals

Name of interval	notation	inequality description	number line representation
finite, open	(a, b)	$a < x < b$	
finite, closed	$[a, b]$	$a \le x \le b$	
finite, half-open	$(a, b]$ $[a, b)$	$a < x \le b$ $a \le x < b$	 
infinite, open	$(a, +\infty)$ $(-\infty, b)$	$a < x < +\infty$ $-\infty < x < b$	 
infinite, closed	$[a, +\infty)$ $(-\infty, b]$	$a \le x < +\infty$ $-\infty < x \le b$	 

Intervals can be used to describe the solution sets of equations and inequalities, the domains of functions, and bounds for estimates. For example, the solution set S of the inequality $x^2 - 6 \ge x$ is the set of all real numbers x with $x \ge 3$ or $x \le -2$, because

$$x^2 - 6 \ge x \iff x^2 - x - 6 \ge 0 \iff (x - 3)(x + 2) \ge 0 \iff x \ge 3 \text{ or } x \le -2.$$

(Read "$\Leftrightarrow$" as "if and only if.") S can be described in interval and in set notation.

$$S = (-\infty, -2] \cup [3, +\infty) = \{x \in \mathbf{R}: x \le -2 \text{ or } x \ge 3\}$$

As another example, the domain D of the tangent function can be described as

$$D = \left\{x \in R: x \ne (2k + 1)\left(\frac{\pi}{2}\right) \quad \text{for all integers } k\right\},$$

or as the union of all open intervals between successive odd multiples of $\frac{\pi}{2}$; that is,

$$D = \bigcup_{k \in Z}\left([2k - 1]\frac{\pi}{2}, [2k + 1]\frac{\pi}{2}\right).$$

As still another example, if the diameter of a rod must be within .05 mm of a desired value of 1.45 cm, then the diameter must lie on the closed interval [1.445 cm, 1.455 cm].

Apart from applications such as these, intervals are also used to define decimal representation and to describe some of the most basic properties of the real number system.

What is a decimal?

We can classify decimals into two categories: finite (terminating) decimals such as 2.25 and 0.4 and infinite (nonterminating) decimals such as $4.2\overline{3}$ or the decimal representation of $\pi = 3.141592\ldots$. We first define finite decimals. To formulate a proper definition, think about how you have used decimals in the past. A real number such as $3\frac{5}{8}$ is written in decimal notation as 3.625 because $3\frac{5}{8} = 3 + \frac{6}{10^1} + \frac{2}{10^2} + \frac{5}{10^3}$. The number 3.625 is a *finite decimal* representing $3\frac{5}{8}$. Here is a general definition.

Definition If a nonnegative real number x can be expressed as a (finite) sum of the form $x = D + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_k}{10^k} = D + d_1 \cdot 10^{-1} + d_2 \cdot 10^{-2} + \cdots + d_k \cdot 10^{-k}$, where D and each d_n are nonnegative integers and $0 \le d_n \le 9$ for $n = 1, 2, \ldots, k$, then $D.d_1d_2\ldots d_k$ is the **finite decimal** representing x.

We say that D is the **integer part**, $.d_1d_2\ldots d_k$ is the **decimal part**, and d_i is the ***i*th decimal place** of the decimal.[4] The integer part is the greatest integer less than or equal to x.

If x is a negative real number and there is a finite decimal $D.d_1d_2\ldots d_k$ representing $-x$, then we write $-(D.d_1d_2\ldots d_k)$ or, more simply, $-D.d_1d_2\ldots d_k$ for the finite decimal representing x. In this case, the integer part is $-D - 1$. For example, if $x = -\frac{11}{8}$, then $x = -(1 + \frac{3}{8}) = -(1.375) = -(1 + \frac{3}{10} + \frac{7}{10^2} + \frac{5}{10^3})$. Thus -1.375 is the finite decimal representing x, with integer part -2, because $D = 1$ and so $-D - 1 = -2$.

Question 1: Give the values of D and the d_i for $98\frac{43}{3200}$.

Question 2:

a. Explain why every finite decimal represents a rational number.

b. Show by example that there are rational numbers that do not have finite decimal representations.

An **infinite decimal** is an infinite sequence $d = [D, d_1, d_2, d_3, \ldots, d_n, \ldots]$ of integers such that $0 \le d_k \le 9$ for all k in N. An infinite decimal d is usually written in the form

$$d = D.d_1d_2d_3\ldots d_n\ldots.$$

Every finite decimal $D.d_1d_2\ldots d_k$ can be regarded as an infinite decimal by identifying it with the infinite sequence $d = [D, d_1, d_2, \ldots, d_k, 0, 0, 0, \ldots, 0, \ldots]$. (See Problem 2 at the end of this section.)

How does a decimal determine a real number?

Think of the decimal for the number π, the circumference of a circle with unit diameter. If only the first two decimal places are known, that is, $\pi = 3.14\ldots$, then we know that π is in the closed interval $[3.14, 3.15]$, an interval of length 10^{-2}. Each succeeding decimal place places π in an interval of length $\frac{1}{10}$ the preceding interval. To five decimal places, $\pi = 3.14159\ldots$, which places π in the interval $[3.14159, 3.14160]$, an interval of length 10^{-5}. This interval is a subset of the preceding intervals and is said to be *nested* in each preceding interval.

[4] Historically, x was called a *decimal fraction*. Most books today avoid this vocabulary because the words "decimal" and "fraction" refer to representations of the number, not the number itself.

Definitions An interval I is **nested** in another interval J if and only if I is a subset of J, that is, $I \subseteq J$. A sequence $\{I_k\}$ of intervals is called a **nested sequence** if and only if $I_{k+1} \subseteq I_k$ for all k.

The determination of a single real number follows from the *Nested Interval Property* of the real numbers, which we assume.

Nested Interval Property: For any sequence of finite closed nested intervals, there is at least one point that belongs to all of them.

The Nested Interval Property is an assumed geometric property of the number line model of the real number system. From it, many important properties of the real numbers and real functions can be derived. These include the identification of the real number determined by a given decimal and rational approximations of real numbers. For instance, consider the nested sequence of closed intervals:

$$[3.1, 3.2], [3.14, 3.15], [3.141, 3.142], [3.1415, 3.1416], \ldots, [p_k, p_k + 10^{-k}], \ldots,$$

where p_k is the rational number whose finite decimal representation consists of the first k places of the decimal $\pi = 3.141592\ldots$. The Nested Interval Property asserts that there is at least one point that belongs to all of these intervals. Moreover, because the length of the kth interval in this sequence is 10^{-k}, at most one point can belong to all of these intervals. (If there were two points, the distance between them would be larger than 10^{-k} for some k, so they could not both be in all the $[p_k, p_k + 10^{-k}]$.) That unique point is the real number π.

More generally, if $\{I_k\}$ is any nested sequence of closed intervals with rational endpoints whose length decreases to 0, there is one and only one point that belongs to all of the intervals in the sequence. That point is the **real number determined by the sequence** $\{I_k\}$. There are many different nested sequence of closed intervals with rational endpoints that determine the same real number. For example, the nested sequences $\{I_k\}$ and $\{J_k\}$ defined by

$$I_k = [p_k - 10^{-k}, p_k + 10^{-k}] \quad \text{and} \quad J_k = \left[p_k, p_k + \frac{1}{k}\right]$$

also determine the real number π. The important thing about nested sequences of closed intervals with rational endpoints with lengths decreasing to 0 is that each such sequence corresponds to exactly one real number, and that number is represented on the number line by the one and only point common to all of the intervals.

Notice that every rational number x is a real number determined by such a sequence, for we need only take $I_n = [x, x + 10^{-n}]$ to obtain a sequence $\{I_k\}$ of nested closed intervals with rational endpoints that all contain x and only x.

Now consider the decimal $D.d_1d_2d_3\ldots d_n\ldots$ (finite or infinite). For each natural number k, define x_k to be the rational number represented by the finite decimal $D.d_1d_2d_3\ldots d_k$, and let $I_k = [x_k, x_k + 10^{-k}]$ be the closed interval with left endpoint x_k and length 10^{-k}. The closed intervals I_k are nested. Because the length of I_k is 10^{-k}, and 10^{-k} decreases to 0 as k increases, there is one and only one point x on the number line that belongs to all of the intervals I_k. This unique x is the **real number determined by the decimal** $D.d_1d_2d_3\ldots d_n\ldots$, and we say that $D.d_1d_2d_3\ldots d_n\ldots$ is a **decimal representation** of x.

EXAMPLE 1 Find the first five terms of a nested sequence $\{I_k\}$ of intervals for the finite decimal 3.625.

Solution Since we are given the decimal 3.625, the first five intervals of $\{I_k\}$ can be found merely by examining 3.625. They are

$$[3.6, 3.7], [3.62, 3.63], [3.625, 3.626], [3.6250, 3.6251], [3.62500, 3.62501].$$

The left endpoint of all of these intervals from I_3 onward is $3\frac{5}{8}$, which is the real number represented by a finite decimal 3.625.

EXAMPLE 2 Find the first five terms of the nested sequence $\{I_k\}$ of intervals for the decimal for $\sqrt{2}$.

Solution We know that $1 < \sqrt{2} < 2$. The first decimal place is the number d_1 from 0 to 9 such that

$$1 + \frac{d_1}{10} \le \sqrt{2} \le 1 + \frac{d_1 + 1}{10}.$$

Since d_1 can only be one of ten values, we can just try them all, and test by squaring the numbers. We find

$$1 + \frac{4}{10} \le \sqrt{2} \le 1 + \frac{5}{10}.$$

In decimal notation,

$$1.4 \le \sqrt{2} \le 1.5.$$

We proceed in the same way to find the second decimal place d_2, which satisfies

$$1 + \frac{4}{10} + \frac{d_2}{100} \le \sqrt{2} \le 1 + \frac{4}{10} + \frac{d_2 + 1}{100}.$$

Again there are only ten possible values and we find $d_2 = 1$ because

$$1 + \frac{4}{10} + \frac{1}{100} \le \sqrt{2} \le 1 + \frac{4}{10} + \frac{2}{100}.$$

That is,

$$1.41 \le \sqrt{2} \le 1.42.$$

Thus, for the decimal d representing $\sqrt{2}$, the first two intervals of $\{I_k\}$ are $[1.4, 1.5]$ and $[1.41, 1.42]$. With a calculator, we can find the next three intervals. They are $[1.414, 1.415]$, $[1.4142, 1.4143]$, and $[1.41421, 1.41422]$.

Question 3: Explain why the real number $\sqrt{2}$ is not an endpoint (left or right) of any of the intervals in the sequence $\{I_k\}$ whose first 5 terms are listed in Example 2.

The method of Example 2 enables us to find a decimal for any real number x that can be compared to rational numbers. Suppose $x > 0$, and let $D = \lfloor x \rfloor$. In Problem 7, you are asked to use the preceding constructions of d_1 and d_2 as the basis to prove that for each natural number k there is an integer d_k such that $0 \le d_k \le 9$ and

$$D + \frac{d_1}{10} + \cdots + \frac{d_k}{10^k} \le x < D + \frac{d_1}{10} + \cdots + \frac{d_k}{10^k} + \frac{1}{10^k}.$$

Then $D.d_1d_2d_3\ldots d_k$ gives an approximation to the real number x **to k decimal places.**

If the given real number x is negative, then apply the preceding construction to the positive real number $y = -x$ to construct a decimal $D.d_1d_2d_3\ldots d_k\ldots$ corresponding to y. Then $-D.d_1d_2d_3\ldots d_k\ldots$ is defined to be the decimal representing x.

Can one real number have two different decimal representations?

The answer to this question is yes. Two different decimal representations exist for all rational numbers with finite decimals.

For example, the construction of a decimal representation for the number 1 one digit at a time, as in the preceding description, results in the decimal representation 1.0000 . . . 00 . . . . However, .9999 . . . 9 . . . is also a decimal representation of 1 because

$$0 + \frac{9}{10} + \cdots + \frac{9}{10^k} < 1 = 0 + \frac{9}{10} + \cdots + \frac{9}{10^k} + \frac{1}{10^k} \quad \text{for all } k \in \mathbf{N}.$$

(See Problem 6.) Similarly, 3.4999 . . . (with 9s repeating forever) = 3.5000 (with 0s repeating forever). In general, the reason two decimals exist for these numbers is that we have defined $d = [D, d_1, d_2, d_3, \ldots, d_k, \ldots]$ to be a decimal representation of a real number x provided that d and x satisfy the following inequalities:

$$D + \frac{d_1}{10} + \cdots + \frac{d_k}{10^k} \le x \le D + \frac{d_1}{10} + \cdots + \frac{d_k}{10^k} + \frac{1}{10^k} \quad \text{for all } k \in \mathbf{N}.$$

Consequently, if x is a decimal fraction represented by the finite decimal $d = D.d_1d_2\ldots d_m$, then

$$D + \frac{d_1}{10} + \cdots + \frac{d_m}{10^m} = x.$$

But then the infinite decimal $d = D.d_1d_2\ldots(d_m - 1)999\ldots 9\ldots$ also represents x because

$$D + \frac{d_1}{10} + \cdots + \frac{d_m - 1}{10^m} < x = D + \frac{d_1}{10} + \cdots + \frac{d_m - 1}{10^m} + \frac{9}{10^{m+1}} + \cdots + \frac{9}{10^k} + \frac{1}{10^k} \quad \text{for all } k \ge m.$$

An alternate approach to decimal representation that results in a unique decimal representation for each real number is explored in Project 1 for this chapter.

It is possible to define operations of addition and multiplication for real numbers through their representation by sequences of nested closed intervals with rational endpoints. If $\{I_k\}$ and $\{J_k\}$ are the sequences of intervals that determine the real numbers x and y, we form a new sequence of nested intervals whose endpoints are the sums of the corresponding intervals for x and for y. This new sequence will contain a single real number we call the sum. For instance, here are the first five members of the sequences for $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{2} + \sqrt{3}$.

$\sqrt{2}$	$\sqrt{3}$	$\sqrt{2} + \sqrt{3}$
$I_1 = [1.4, 1.5]$	$J_1 = [1.7, 1.8]$	$[3.1, 3.3]$
$I_2 = [1.41, 1.42]$	$J_2 = [1.73, 1.74]$	$[3.14, 3.16]$
$I_3 = [1.414, 1.415]$	$J_3 = [1.732, 1.733]$	$[3.146, 3.148]$
$I_4 = [1.4142, 1.4143]$	$J_4 = [1.7320, 1.7321]$	$[3.1462, 3.1464]$
$I_5 = [1.41421, 1.41422]$	$J_5 = [1.73205, 1.73206]$	$[3.14626, 3.14628]$

Although the intervals of the sum are not the same length as the intervals for $\sqrt{2}$ and $\sqrt{3}$, they are still nested and, since their lengths go to 0, there is only one number within all of them. In similar fashion, we can define multiplication of two real numbers. With these definitions, we can derive all the familiar properties of addition and multiplication of real numbers.

Repeating decimals

The decimal representing any rational number can be obtained by repeated application of the Division Algorithm. We illustrate this process by finding the decimal for $\frac{462}{13}$. The first application gives us 35, the integer part of this decimal. Each succeeding application uses 10 times the remainder from the previous step. We show the first four lines, which yield a quotient of 35.5384 and a remainder of $\frac{8}{130000}$.

$$462 = 13 \cdot 35 + 7$$

$$70 = 13 \cdot 5 + 5 \quad \Rightarrow \quad 7 = 13 \cdot \frac{5}{10} + \frac{5}{10} \quad \Rightarrow \quad 462 = 13\left(35 + \frac{5}{10}\right) + \frac{5}{10}$$

$$50 = 13 \cdot 3 + 11 \quad \Rightarrow \quad \frac{5}{10} = 13 \cdot \frac{3}{10^2} + \frac{11}{10^2} \quad \Rightarrow \quad 462 = 13\left(35 + \frac{5}{10} + \frac{3}{10^2}\right) + \frac{11}{10^2}$$

$$110 = 13 \cdot 8 + 6 \quad \Rightarrow \quad \frac{11}{10^2} = 13 \cdot \frac{8}{10^3} + \frac{6}{10^3} \quad \Rightarrow \quad 462 = 13\left(35 + \frac{5}{10} + \frac{3}{10^2} + \frac{8}{10^3}\right) + \frac{6}{10^3}$$

$$60 = 13 \cdot 4 + 8 \quad \Rightarrow \quad \frac{6}{10^3} = 13 \cdot \frac{4}{10^4} + \frac{8}{10^4} \quad \Rightarrow \quad 462 = 13\left(35 + \frac{5}{10} + \frac{3}{10^2} + \frac{8}{10^3} + \frac{4}{10^4}\right) + \frac{8}{10^4}$$

Because there are only 12 possible nonzero remainders, the cycle of quotients that begins 5384... must repeat after at most 12 steps.

Question 4: Carry out the next two applications of the Division Algorithm to show that the cycle of quotients for $\frac{462}{13}$ repeats after 6 steps.

Long division is a collapsed version of the aforementioned process. Here is the long division to find the decimal for $\frac{462}{13}$. (See Section 5.2.1 for another example relating long division and the Division Algorithm.) Compare each line with the steps of the calculations preceding Question 4. We show the part obtaining the integer 35 as one step even though most people would take two steps to get it. The final remainder 7 is equal to a remainder six steps earlier, so the cycle of quotients, 538461, will be repeated if the long division is continued. Therefore, $\frac{462}{13} = 35.538461538461\ldots538461\ldots$. We write this as $35.\overline{538461}$.

$$\begin{array}{r}
35.538461 \\
13\overline{)462.000000} \\
\underline{455} \\
7\,0 \\
\underline{6\,5} \\
50 \\
\underline{39} \\
110 \\
\underline{104} \\
60 \\
\underline{52} \\
80 \\
\underline{78} \\
20 \\
\underline{13} \\
7
\end{array}$$

More generally, if the decimal representation of a rational number $\frac{a}{b}$ does not terminate, then the decimal is **periodic** (or **repeating**); that is, there is a finite string $d_q d_{q+1} \ldots d_{q+p-1}$ of p digits in the decimal representation of that repeats forever from some point on. This is due to the fact that all of the remainders that occur in the Division Algorithm division procedure for constructing the decimal representation of $\frac{a}{b}$ must be positive integers less than b. Because there are only $b - 1$ such integers, the long division process must eventually cycle.

The shortest repeating string is called a **repetend**, and the length p of a repetend is called the **period** of the decimal. We have demonstrated the following theorem.

Theorem 2.4 Suppose $\frac{a}{b}$ is a rational number in lowest terms with $b > 0$ whose decimal representation is not terminating. Then $\frac{a}{b}$ is represented by a repeating decimal whose period is at most $b - 1$.

Sometimes the period of $\frac{a}{b}$ equals $b - 1$, as for $\frac{1}{7} = .\overline{142857}$. Sometimes the period of $\frac{a}{b}$ does not equal $b - 1$, as for $\frac{3}{11} = .\overline{27}$. In the next section, we explore the period of a periodic decimal representing a rational number.

2.1.2 Problems

1. Find decimals representing the rational numbers $\frac{21}{20}$ and $\frac{20}{21}$.

2. Suppose that x is a rational number represented by the finite decimal $D.d_1d_2 \ldots d_k$.

a. Explain why an (infinite) decimal representing x is $D.d_1d_2 \ldots d_k0000 \ldots$.

b. What other infinite decimal represents x?

3. a. Give the first six digits in the decimal representation of $-\pi$.

b. Describe the first six intervals I_k in the construction of the decimal for $-\pi$.

4. Find the first three decimal places of $\sqrt{7}$ using only multiplication.

5. Find the decimal for each rational number by repeated applications of the Division Algorithm.

a. $\frac{817}{37}$ b. $\frac{6}{25}$ c. $\frac{46}{12}$

6. The following arguments are frequently used to convince students that $.9999\ldots = 1$:

Argument 1: Let $n = .99999\ldots$. Then $10n = 9.99999\ldots$, and so $9n = 10n - n = 9.0000\ldots$; therefore, $9n = 9$ and so $n = 1$.

Argument 2: We know that $\frac{1}{3} = .333333\ldots$, so $1 = 3(\frac{1}{3}) = 3(.33333333\ldots) = .99999999\ldots$.

Argument 3: By long division, you can see that

$$\frac{4}{9} = .44444\ldots \text{ and } \frac{5}{9} = .555555\ldots.$$

Therefore,

$$1 = \frac{4}{9} + \frac{5}{9} = .44444\ldots + .555555\ldots = .99999\ldots.$$

Argument 4: The decimal $.999999\ldots$ stands for the geometric series

$$\frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} + \ldots.$$

But when $|r| < 1$, the sum of the infinite geometric series $1 + r + r^2 + \cdots + r^n + \cdots$ is

$$\frac{1}{1-r}.$$

Therefore,

$$\begin{aligned} .9999\cdots &= \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} + \cdots \\ &= \frac{9}{10}\left(1 + \left(\frac{1}{10}\right) + \left(\frac{1}{10}\right)^2 + \cdots \right. \\ &\qquad \left. + \left(\frac{1}{10}\right)^{n-1} + \cdots\right) \\ &= \frac{9}{10}\left(\frac{1}{1 - \frac{1}{10}}\right) = 1. \end{aligned}$$

Give a justification for each step in these arguments.

7. Suppose that p and q are positive integers and that $a = \frac{p}{q}$. Explain why long division of p by q results in the decimal representation of a. (*Hint*: It is enough to explain why the decimal $d = [D, d_1, d_2, d_3, \dots, d_k, \dots]$ produced by long division satisfies

$$D + \frac{d_1}{10} + \cdots + \frac{d_k}{10^k} \le a < D + \frac{d_1}{10} + \cdots + \frac{d_k}{10^k} + \frac{1}{10^k}$$

for all k in N.)

***8.** Prove that if x is a positive real number, then for each natural number k there is an integer d_k such that $0 \le d_k \le 9$ and such that

$$D + \frac{d_1}{10} + \cdots + \frac{d_k}{10^k} \le x < D + \frac{d_1}{10} + \cdots + \frac{d_k}{10^k} + \frac{1}{10^k}.$$

(*Hint*: Use the construction of d_1 and d_2 in this section as a guide for a proof by mathematical induction.)

ANSWERS TO QUESTIONS

1. $D = 98$, $d_1 = 0$, $d_2 = 1$, $d_3 = 3$, $d_4 = 4$, $d_5 = 3$, $d_6 = 7$, $d_7 = 8$.

2. a. Let $a = D.d_1d_2\dots d_k$. Then $10^k \cdot a$ is an integer, call it n. Then $a = \frac{n}{10^k}$, which is a quotient of integers, so a is rational.

3. $\sqrt{2}$ is not a rational number, unlike all the endpoints, which are rational numbers.

4. $80 = 13 \cdot 6 + 2 \Rightarrow \frac{8}{10^4} = 13 \cdot \frac{6}{10^5} + \frac{2}{10^5} \Rightarrow 462 = 13\left(35 + \frac{5}{10} + \frac{3}{10^2} + \frac{8}{10^3} + \frac{4}{10^4} + \frac{6}{10^5}\right) + \frac{2}{10^5}$;

$20 = 13 \cdot 1 + 7 \Rightarrow \frac{20}{10^5} = 13 \cdot \frac{1}{10^6} + \frac{7}{10^6} \Rightarrow 462 = 13\left(35 + \frac{5}{10} + \frac{3}{10^2} + \frac{8}{10^3} + \frac{4}{10^4} + \frac{6}{10^5} + \frac{1}{10^6}\right) + \frac{7}{10^6}$.

Table 2

$\frac{1}{2} = 0.5$
$\frac{1}{3} = 0.\overline{3}$
$\frac{1}{4} = 0.25$
$\frac{1}{5} = 0.2$
$\frac{1}{6} = 0.1\overline{6}$
$\frac{1}{7} = 0.\overline{142857}$
$\frac{1}{8} = 0.125$
$\frac{1}{9} = 0.\overline{1}$
$\frac{1}{10} = 0.1$
$\frac{1}{11} = 0.\overline{09}$
$\frac{1}{12} = 0.08\overline{3}$
$\frac{1}{13} = 0.\overline{076923}$
$\frac{1}{14} = 0.0\overline{714285}$
$\frac{1}{15} = 0.0\overline{6}$
$\frac{1}{16} = 0.0625$

2.1.3 Periods of periodic decimals

The variety of types of decimals for rational numbers is illustrated in Table 2, which shows the decimal representations of the reciprocals of the integers 2 through 16.

Notice that six of these representations are finite decimals, while the remaining nine are periodic: Five have period 1, one has period 2, and three have period 6.

Also, for five of the nine periodic cases, the period starts right after the decimal point. In three of the other four, there is a delay of 1 digit before the period starts. In one periodic case, there is a delay of 2 digits.

What kind of pattern is there in the types of representations? More precisely, given an integer n, what can you predict about the decimal representation of $\frac{m}{n}$ if m and n are relatively prime positive integers?

The three types of decimals

Theorem 2.4 in the preceding section tells us that, if we divide 1 by an integer n, then after at most $n - 1$ steps the division process must either terminate or else start to repeat. This is true because there are only $n - 1$ possible nonzero remainders. If the process starts to repeat at some step, it can either start repeating from the beginning of the division process, or else from some intermediate point. This gives rise to three distinct types of decimal representations. Table 3 classifies the decimals from Table 2 into these types.

Table 3

Type of Decimal	Examples	General Form
terminating	0.5, 0.25, 0.2, 0.125, 0.1, 0.0625	$0.d_1d_2d_3\dots d_t$ $(d_t \ne 0)$
simple-periodic	$0.\overline{3}$, $0.\overline{142857}$, $0.\overline{1}$, $0.\overline{09}$, $0.\overline{076923}$	$0.\overline{d_1d_2d_3\dots d_p}$
delayed-periodic	$0.1\overline{6}$, $0.08\overline{3}$, $0.0\overline{714285}$, $0.0\overline{6}$	$0.d_1d_2d_3\dots d_t\overline{d_{t+1}d_{t+2}d_{t+3}\dots d_{t+p}}$

* An asterisk by a problem number indicates a problem perceived to be more difficult than other problems.

Question 1: Find the decimal representation of $\frac{1}{28}$ using the Division Algorithm or long division. Into which of the three categories does it fall? Explain the length of the repetend and the number of digits of the delay before the period starts in terms of the pattern of the remainders.

The examples in Tables 2 and 3 are all decimal representations of reciprocals of positive integers $\frac{1}{n}$. However, it is easy to extend what we learn from these reciprocals to decimal representations of any positive number $\frac{m}{n}$, where m and n are positive integers and $m < n$ because of the following result: *If $\frac{m}{n}$ is in lowest terms, the general form of the decimal representation of $\frac{m}{n}$ is the same as that of $\frac{1}{n}$.* (See Problem 2.)

By the *general form* of a decimal representation, we mean not only the type (terminating, simple-periodic, delayed-periodic), but also the length of the strings of digits in each part (the values of t and p in the notation of the second column of Table 3). The proof of this result will become apparent as we work through each type.

The big picture

Table 4 summarizes the results that are deduced in the remainder of this section. We need only consider rational numbers x between 0 and 1 written as $x = \frac{m}{n}$ in lowest terms because any other rational number is the sum of x and an integer, and the integer part of a rational number does not affect the general form of its decimal representation.

Table 4

Type of Decimal Representation	Form of Decimal Representation[5]	Rational Number $\frac{m}{n}$ in Lowest Terms	Equivalent Form of Rational Number
terminating	$0.d_1d_2d_3\ldots d_t$ (t = max of r and s)	$\frac{m}{2^r \cdot 5^s}$ (Theorem 2.6)	$\frac{M}{10^t}$ (Theorem 2.5)
simple-periodic	$0.\overline{d_1d_2d_3\ldots d_p}$	$\frac{m}{3^u \cdot 7^v \cdot 11^w \cdot \ldots}$ (Theorem 2.8)	$\frac{M}{10^p - 1}$ (Theorem 2.7)
delayed-periodic	$0.d_1d_2\ldots d_t\overline{d_{t+1}d_{t+2}\ldots d_{t+p}}$ (t = max of r and s, $t > 0$)	$\frac{m}{2^r \cdot 5^s \cdot 3^u \cdot 7^v \cdot 11^w \cdot \ldots}$ (Theorem 2.10)	$\frac{M}{10^t \cdot (10^p - 1)}$ (Theorem 2.9)

Terminating decimals

We begin with the simplest case: terminating decimals.

Suppose $x = \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_t}{10^t}$ where $d_1, d_2, \ldots, d_t$ are nonnegative integers, $d_t \neq 0$, and $d_k \leq 9$ for $k = 1, \ldots, t$. Any such number is represented by the terminating decimal $.d_1d_2\ldots d_{t-1}d_t$ with $d_t \neq 0$. Then $x = \frac{M}{10^t}$, where

$$M = 10^{t-1}d_1 + 10^{t-2}d_2 + \ldots 10d_{t-1} + d_t.$$

For example, because $\frac{5}{8} = \frac{6}{10} + \frac{2}{10^2} + \frac{5}{10^3}$, $\frac{5}{8}$ is represented by the terminating decimal .625. Also, $\frac{5}{8}$ can be expressed as $\frac{5}{8} = \frac{625}{10^3}$, and

$$625 = 10^2 \cdot 6 + 10^1 \cdot 2 + 5.$$

[5]Throughout this section, strings of d_i such as $d_1d_2d_3\ldots d_t$ stand for digits of a number (and not for multiplication).

Conversely, a fraction with a denominator that is a power of 10 has a terminating decimal representation that can be immediately written. For instance, $\frac{902}{10^8} = \frac{9 \cdot 10^2 + 2}{10^8} = \frac{9}{10^6} + \frac{2}{10^8} = .00000902$. You are asked to show this converse (see Problem 9).

Theorem 2.5 A number x between 0 and 1 has a terminating decimal representation $0.d_1d_2d_3 \ldots d_t$ (where $d_t \neq 0$) if and only if it can be represented in the form $x = \frac{M}{10^t}$ for some positive integer M that is not divisible by 10.

To use Theorem 2.5, we need to know whether or not a given rational number $\frac{m}{n}$ can be represented in the form $\frac{M}{10^t}$. For example, $\frac{1}{250}$ and $\frac{1}{64}$ can be represented in this form, but $\frac{1}{60}$ cannot.

Question 2: Explain why $\frac{1}{250}$ and $\frac{1}{64}$ have terminating decimals but $\frac{1}{60}$ does not.

There is a simple way of telling these cases apart.

Theorem 2.6 Suppose that $\frac{m}{n}$ is in lowest terms and that $m < n$. Then $\frac{m}{n}$ has a decimal representation that terminates after t digits if and only if $n = 2^r \cdot 5^s$ and t is the larger of r and s.

Proof: Since the theorem is an if-and-only-if statement, both directions of the implication must be proved.

($\Rightarrow$) Suppose $\frac{m}{n}$ has a decimal representation $D.d_1d_2d_3 \ldots d_t$. Then, by Theorem 2.5, $\frac{m}{n}$ can be represented in the form $\frac{M}{10^t} = \frac{M}{2^t \cdot 5^t}$ where M is not divisible by 10. Thus M may contain factors of 2 or 5 but not both. If we cancel these factors in M with corresponding factors in the denominator, we obtain a denominator $n = 2^r \cdot 5^s$, where t is the larger of r and s.

($\Leftarrow$) Suppose $n = 2^r \cdot 5^s$ and $r \geq s$. Then we can write $r = s + k$ for some non-negative integer k, and so $\frac{m}{n} = \frac{m}{2^{s+k}5^s} = \frac{m \cdot 5^k}{2^{s+k}5^{s+k}} = \frac{m \cdot 5^k}{10^r}$, and $m \cdot 5^k$ is not divisible by 10 because $\frac{m}{n}$ is in lowest terms and n has factor of 2. This shows that the decimal representation of $\frac{m}{n}$ consists of the decimal representation of the integer $m \cdot 5^k$, but with the decimal point moved r digits to the left. The case $r < s$ is similar. ∎

Question 3: What fraction $\frac{m}{n}$ in lowest terms is represented by the terminating decimal 0.00056?

Question 4: Use Theorem 2.6 to find the number of digits in the decimal representation for $\frac{7}{80}$.

Decimals with simple-periodic representations

Theorem 2.6 shows that terminating decimals represent rational numbers in lowest terms whose denominators consist *only* of powers of 2 and 5. Now, we consider the decimal representations of those rational numbers in lowest terms whose denominators contain *no* powers of 2 and 5. Examples from Table 2 are written here.

$$\frac{1}{3} = 0.\overline{3} \qquad \frac{1}{7} = 0.\overline{142857} \qquad \frac{1}{9} = 0.\overline{1} \qquad \frac{1}{11} = 0.\overline{09} \qquad \frac{1}{13} = 0.\overline{076923}$$

It happens that these numbers all have *simple-periodic* decimal representations. (The periods for the five numbers just given are 1, 6, 1, 2, and 6, respectively.) Although simple to state, this property is not so easy to prove. We first demonstrate that *every rational $\frac{m}{n}$ that is represented by a simple-periodic decimal is equal to a fraction with a denominator consisting of all 9s*. For example, for the simple-periodic decimals from Table 2,

$$0.\overline{3} = \frac{3}{9} \qquad 0.\overline{1} = \frac{1}{9} \qquad 0.\overline{09} = \frac{9}{99} \qquad 0.\overline{076923} = \frac{76923}{999999} \qquad 0.\overline{142857} = \frac{142857}{999999}.$$

Notice not only that these simple-periodic decimals represent fractions with all 9s in the denominator, but also that the length of the period tells us how many 9s there are. It is easy to prove this.

Proof: Suppose $x = 0.\overline{d_1d_2d_3\ldots d_p}$. Multiply x by 10^p. Then subtract x to obtain $(10^p - 1)x = d_1d_2\ldots d_p$, a finite decimal. Then divide by $10^p - 1$ to obtain $x = \frac{d_1d_2d_3\ldots d_p}{10^p - 1}$; that is, x can be expressed as a fraction in which the numerator is the repetend of x and the denominator is the integer with p digits all equal to 9. ∎

Question 5: Represent $.\overline{314}$ in this form.

We have proved the ($\Rightarrow$) direction of a statement whose converse is also true. That is, if a fraction can be written in a form (not necessarily lowest terms) with a denominator consisting of a string of 9s, then its decimal representation is simple-periodic. You can get a feel for this fact by working with actual examples using a calculator. The proof of the general statement is not difficult.

Theorem 2.7 A number x between 0 and 1 has a simple-periodic decimal representation $0.\overline{d_1d_2d_3\ldots d_p}$ if and only if x can be put in the form $\frac{M}{10^p - 1}$, where M is the integer $d_1d_2d_3\ldots d_p$.

Proof: Having proved the ($\Rightarrow$) direction, we show the proof of the ($\Leftarrow$) direction. Suppose $x = \frac{M}{10^p - 1}$ and M is the integer $d_1d_2d_3\ldots d_p$. Now let $\overline{M}$ be the periodic decimal $.\overline{d_1d_2d_3\ldots d_p}$. Then $10^p\overline{M} = M + \overline{M}$. So $\overline{M} = \frac{M}{10^p - 1}$. Consequently $\overline{M} = x$. ∎

Question 6: In the representation form guaranteed by this theorem, are M and $10^p - 1$ always relatively prime?

How do we apply Theorem 2.7? For example, given a rational number such as $\frac{1}{13}$, how do we know whether it can be put in a form $\frac{M}{10^p - 1}$ with all 9s in the denominator? We now are ready to prove the test we mentioned earlier.

Theorem 2.8 Suppose that $\frac{m}{n}$ is in lowest terms and that $m < n$. Then $\frac{m}{n}$ has a simple-periodic decimal representation if and only if 2 or 5 are not factors of n.

Proof:

($\Rightarrow$) If the rational number $\frac{m}{n}$ has a simple-periodic decimal representation, then by Theorem 2.7 it can be put in the form $\frac{M}{10^p - 1}$, and so $m \cdot (10^p - 1) = M \cdot n$. If n were to have a factor of 2 or 5, then m would also have that factor, since clearly $10^p - 1$ cannot. But this would contradict the fact that $\frac{m}{n}$ is in lowest terms. We conclude that n can have no factor of 2 or 5.

($\Leftarrow$) To prove the other direction of the if-and-only-if statement, suppose 2 and 5 are not factors of n. Then, by Theorem 2.6, $\frac{m}{n}$ cannot have a terminating decimal representation, so it will have a periodic decimal representation with a period p.

Consider the remainders r_k that occur in the long division of m by n with the numerator m regarded as the initial remainder r_0. We know that for some smallest nonnegative t, it is true that $r_{k+p} = r_k$ for all $k \geq t$. We must show that $t = 0$; that is, that the remainders begin to repeat after the first p steps.

Suppose to the contrary that $t > 0$. Then $r_{t+p} = r_t$ but $r_{t+p-1} \neq r_{t-1}$. The Division Algorithm implies that there exist integers q_t, q_{t+p}, r_t, and r_{t+p} with $0 < r_t < n$ and $0 < r_{t+p} < n$ such that

$$10r_{t-1} = nq_t + r_t$$
$$10r_{t+p-1} = nq_{t+p} + r_{t+p}$$

for suitable nonnegative integers q_t and q_{t+p}. By subtracting the first of these equations from the second and using the fact that $r_{t+p} = r_t$, we conclude that

$$10(r_{t+p-1} - r_{t-1}) = n(q_{t+p} - q_t).$$

Therefore, n is a divisor of $10(r_{t+p-1} - r_{t-1})$. Because n has no factors of 2 or 5, n must be a divisor of $r_{t+p-1} - r_{t-1}$. But the remainders r_{t+p-1} and r_{t-1} are positive integers less than n, so their difference $r_{t+p-1} - r_{t-1}$ is an integer between $-n$ and n. Because n is also a divisor of this integer, this integer must be 0. Thus, $r_{t+p-1} = r_{t-1}$, contrary to our supposition that $t > 0$. Therefore, the decimal representation of $\frac{m}{n}$ must be simple-periodic. ┘

Decimals with delayed-periodic representations

The characteristics of delayed-periodic decimals are a combination of the characteristics of the two previous cases (terminating decimals and simple-periodic decimals). Delayed-periodic decimals result when the denominator of $\frac{m}{n}$ in lowest terms has both 2's or 5's and some other prime factors as well. The highest power of the 2 or the 5 that divides n determines the length t of the delay before the repetend starts. But the period itself is determined only from an equal fraction that is found in basically the same way as for simple-periodic decimals.

Question 7: Consider $\frac{9}{28}$.

a. Give its decimal representation.

b. What is the length t of the delay?

c. What is the period p?

d. Express $\frac{9}{28}$ as a sum of the form $\frac{A}{10^t} + 10^{-t}\frac{B}{10^p - 1}$, where A and B are integers.

Theorem 2.9 A number x between 0 and 1 has a delayed-periodic decimal representation $0.d_1d_2\ldots d_t\overline{d_{t+1}d_{t+2}\ldots d_{t+p}}$ with period p and with t digits before the start of the repetend if and only if $x = \frac{M}{10^t(10^p - 1)}$, where the numerator is the integer $M = d_1d_2\ldots d_td_{t+1}d_{t+2}\ldots d_{t+p} - d_1d_2\ldots d_t$. (The denominator has a decimal representation consisting of p 9s followed by t 0s.)

Proof: Suppose x has the delayed-periodic decimal representation of the theorem. Multiply x by $10^t \cdot 10^p$ to obtain

$$10^t \cdot 10^p \cdot x = d_1d_2\ldots d_td_{t+1}d_{t+2}\ldots d_{t+p}.\overline{d_{t+1}d_{t+2}\ldots d_{t+p}}.$$

Then $10^t \cdot 10^p \cdot x - 10^t \cdot x = d_1d_2\ldots d_td_{t+1}d_{t+2}\ldots d_{t+p} - d_1d_2\ldots d_t = M$, where M is an integer. Solve for x to obtain the desired result: $x = \frac{M}{10^t(10^p - 1)}$.

To prove the other direction of the theorem, as the answer to Question 7 illustrates, $x = \frac{M}{10^t(10^p - 1)}$ can also be represented as a sum,

$$\frac{M}{10^t(10^p - 1)} = \frac{A}{10^t} + 10^{-t}\frac{B}{10^p - 1},$$

for suitable constants A and B. The result follows from this fact. The details are left for you. ⌋

Theorem 2.9 gives a necessary and sufficient condition for a fraction to have a delayed-periodic decimal representation.

Finally, we give the simple condition for predicting from the denominator n of a rational number $\frac{m}{n}$ in lowest terms those that have delayed-periodic forms. The proof follows from our earlier results.

Theorem 2.10 A rational number $\frac{m}{n}$ between 0 and 1 that is in lowest terms has a delayed-periodic decimal representation with a period that starts t digits after the decimal point if and only if $n = 2^r \cdot 5^s \cdot q$ for some integer q relatively prime to 2 and 5. Here r and s are not both 0, and t is the larger of r and s.

You might find it helpful to refer back to Table 4, which provides a summary of this section. Notice that the type of decimal representation for a rational number $\frac{m}{n}$ in lowest terms is determined entirely by the factors of n (see Problem 2).

2.1.3 Problems

1. Use the theorems of the text to predict the general form of the decimal representations of the reciprocals of the integers 17 to 41; that is, the type as well as the period and delay if the type is periodic.

2. Explain why, if $\frac{m}{n}$ is a rational number between 0 and 1, and if m and n are relatively prime, then the type of the decimal representation of $\frac{m}{n}$ is independent of m. (*Hint*: Consider the three types of decimal representations separately.)

3. a. The rational number $\frac{1}{19}$ has decimal representation $0.\overline{052631578947368421}$. This can be verified by a pencil and paper application of the Division Algorithm, but this is laborious. It would be nice to be able to do this on a calculator, but many calculators show only 6 or 7 digits. Find a method for using a calculator to "piece together" the full 18 digits of the period of $\frac{1}{19}$.

b. Find the decimal representation of $\frac{12}{71}$.

4. Consider those *reciprocals of primes* that have *simple-periodic* decimal representations. Using the theorems of the section, prove that, of these:

a. There is exactly 1 with period 1. What is it?

b. There is exactly 1 with period 2. What is it?

c. There is exactly 1 with period 3. What is it?

d. There is exactly 1 with period 4. What is it?

e. There are exactly 2 with period 5. What are they?

f. There are exactly 2 with period 6. What are they?

5. Consider those *reciprocals of integers* that have *simple-periodic* decimal representations. Using the theorems of the section, prove that, of these:

a. There are exactly 2 with period 1. What are they?

b. There are exactly 3 with period 2. What are they?

c. There are exactly 5 with period 3. What are they?

6. a. Find the decimal representations for $\frac{1}{27}$ and $\frac{1}{37}$.

b. Explain the peculiar relationship between these decimals, and find other pairs of decimals with the same relationship.

7. Find the prime factorization of integers of the form $10^p - 1$ for $p = 1$ to 7. What is the relevance of these factorizations to the behavior of decimal representations?

8. Consider the decimal representations for $\frac{n}{7}$:

$$\frac{1}{7} = 0.\overline{142857}, \qquad \frac{2}{7} = 0.\overline{285714}, \qquad \text{etc.}$$

Find the other four representations. Notice that the digits of the period always appear in the same order. Is this peculiar to the denominator 7, or is it true of all prime denominators?

9. Prove the ($\Leftarrow$) direction of Theorem 2.5: If $x = \frac{M}{10^t}$ and M is an integer not divisible by 10, then x has a terminating decimal representation.

10. Another proof of the ($\Leftarrow$) implication for Theorem 2.8 is based on a theorem proved by Euler. Euler's theorem, which is a generalization of Fermat's Theorem (Section 5.3.2), states that if natural numbers a and n are relatively prime (have no common factors larger than 1), then the number $a^{\phi(n)} - 1$ has n as a factor. Here, $\phi(n)$ is the **Euler phi function**, defined as the number of natural numbers less than n and relatively prime to n. Since 10 and any number n with no factors of 2 or 5 are relatively prime, the theorem guarantees that $(10^{\phi(n)} - 1) = n \cdot M$ for some integer M. Hence we can write

$$\frac{m}{n} = \frac{m \cdot M}{n \cdot M} = \frac{m \cdot M}{10^{\phi(n)} - 1}$$

So Euler's Theorem provides the denominator of 9s. Theorem 2.7 now applies, showing that $\frac{m}{n}$ has a simple-periodic decimal representation.

Prove the following: The period p of the decimal representation of a rational number $\frac{m}{n}$ in lowest terms is a divisor of the number $\phi(n)$ of positive integers less than n that are relatively prime to n.

11. Consider $\frac{1}{21}$. You may need to refer to Problem 10.

a. Use the theorems of this section to explain why its decimal representation must be simple-periodic.

b. By computing $\phi(21)$, show that the period p of its decimal representation must divide 12, and hence be equal to 1, 2, 3, 4, 6, or 12.

c. Show that 21 must thus divide one of the numbers 99, 999, 9999, 999999, or 999999999999.

d. Show that the first of these numbers that 21 does divide is $10^6 - 1 = 999999$, and conclude that the period must be 6.

e. Compute the decimal for $\frac{1}{21}$ to verify your result.

12. Finish the proof of Theorem 2.9.

13. Prove Theorem 2.10.

ANSWERS TO QUESTIONS

1. $\frac{1}{28} = 0.03\overline{571428}$; delayed-periodic; remainders are 10, 16, 20, 4, 12, 8, 24, 16. The second appearance of 16 indicates a repetend 6 digits long; the first two quotients are 0 and 3, indicating a delay of 2 digits before the repetend.

2. $250 = 2^1 \cdot 5^3$, and $64 = 2^6 \cdot 5^0$, but $60 = 3 \cdot 2^2 \cdot 5^1$, which is not a product of 2's and 5's.

3. $\frac{56}{10^5} = \frac{7}{12500} = \frac{7}{2^2 5^5}$.

4. $\frac{7}{80} = \frac{7}{2^4 5^1}$. Hence $r = 4$, $s = 1$, and so $t = 4$. This means there will be 4 digits in the decimal representation of $\frac{7}{80}$. In fact, $\frac{7}{80} = 0.0875$.

5. $\frac{314}{999}$.

6. No. For example, the representation of this form for $0.\overline{3}$ is $\frac{3}{9}$.

7. a. $\frac{9}{28} = 0.32\overline{142857}$

b. 2

c. 6

d. $\frac{9}{28} = \frac{32}{10^2} + 10^{-2}\left(\frac{142857}{10^6 - 1}\right)$

2.1.4 The distributions of various types of real numbers

In this section, we consider the following question: How are the rational numbers and irrational numbers distributed among the real numbers? To answer this question, we first note that every real number is either an integer or lies between two integers.

Theorem 2.11 For any real number x, there is one and only one integer n such that $n \le x < n + 1$.

Proof: This number n is the greatest integer less than or equal to x, $\lfloor x \rfloor$. If a decimal representation of x is $D.d_1d_2d_3\ldots$, then $\lfloor x \rfloor = D$ if x is positive, and $\lfloor x \rfloor = D - 1$ if x is negative. ┘

The integers are sprinkled through the real numbers in the sense that it is easy to find two real numbers between which there is no integer. But even if there is no integer between two real numbers x and y, there always is a rational number and an irrational number between them.

Theorem 2.12 For any pair of real numbers x and y with $x < y$, there is a rational number r and an irrational number s such that r and s are both in the open interval (x, y).