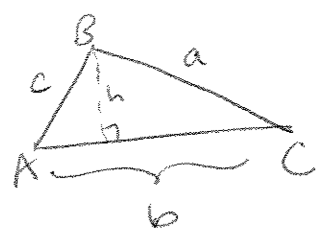
6.1 Law of Sines

You can use the Law of Sines to solve for info in a "non-right" triangle. There are six pieces of info for any triangle, namely the 3 leg lengths and the 3 angles.

Law of Sines

For triangle ABC with sides a, b & c

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

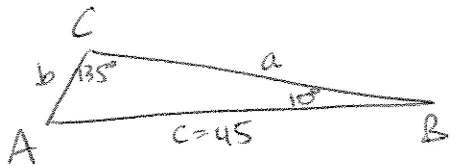


Use when we have these cases:

① ASA (or AAS)

② SSA (ambiguous case) ☆

Ex 1 Use law of Sines to solve the triangle

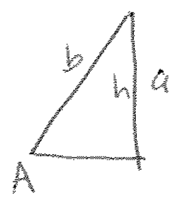


6.1 (cont)

SSA case (ambiguous case)

Assume you're given a, b and A. (Note: $h = b\sin A$

)

						
condition	A acute $a<h$	A acute $a=h$ (rt. Δ)	A acute $a>b$	A acute $h<a<b$	A obtuse $a\le b$	A obtuse $a>b$
# Δs possible	0	1	1	2	0	1

Ex 2 How many Δs are possible?

(a) $A=62^\circ$, $a=10$, $b=12$

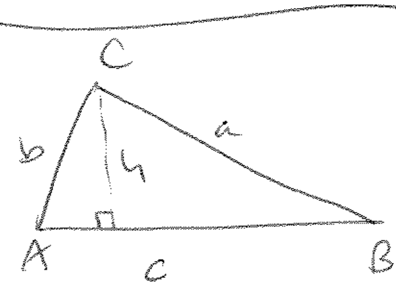
(b) $A=98^\circ$, $a=10$, $b=3$

(c) $A=54^\circ$, $a=7$, $b=10$

6.1 (cont)

Ex 3 Find the triangle info given $c = 29$, $b = 46$ and $C = 31°$.

Area of Triangle



$\text{Area} = \frac{1}{2}(\text{base})(\text{height})$

$\text{Area} = \frac{1}{2} ch$

but $\frac{h}{b} = \sin A \Rightarrow h = b \sin A$

$\Rightarrow \text{Area} = \frac{1}{2} cb \sin A$

Equivalently, $\boxed{\text{Area} = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B}$

6.1 (cont)

Ex 4 Find the area of the triangle w/ $B = 120°$, $a = 32$ and $c = 50$.

6.2 Law of Cosines

Cases that are left (that Law of Sines didn't cover): ③ SSS and ④ SAS

Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos A$ OR $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

$b^2 = a^2 + c^2 - 2ac\cos B$ OR $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$

$c^2 = a^2 + b^2 - 2ab\cos C$ OR $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

Use when we have these cases:
③ SSS
or ④ SAS

(Note: If we have a right triangle, and we know 2 leg lengths, then that is the SAS case + we can use the Law of Cosines, which then simplifies to Pythagorean Thm! ☺)

Ex 1 (more space on next page)
Find the missing information for $\triangle ABC$, given $a = 24$, $b = 16$ and $c = 29$.

(Note: Solve for biggest angle first.)

6.2 (cont)

Ex 1 (cont)

Ex 2 On a baseball diamond w/ 90-foot sides, the pitcher's mound is 60.5 ft from home plate. How far is it from the pitcher's mound to third base?

6.2 (cont)

Ex 3 Law of Sines or Law of Cosines?

(a) $A = 15°$, $B = 58°$, $c = 94$

(b) $a = 96$, $b = 43$, $A = 105°$

(c) $a = 23$, $b = 19$, $c = 15$

(d) $a = 15$, $c = 42$, $B = 49°$

Heron's Area Formula

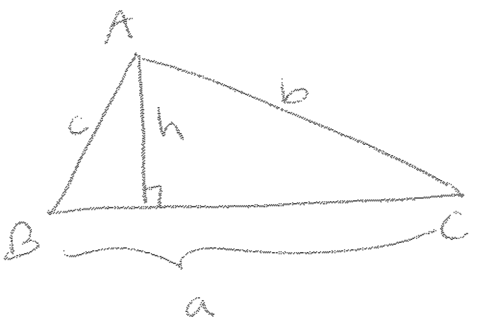
Given any triangle w/ sides of length a, b and c, the area of the triangle is

$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{given } S = \frac{a+b+c}{2}$$

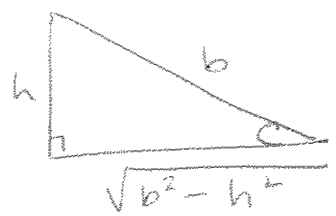
6.2 (cont)

Ex 4 Find the area of the ΔABC such that $a = 12$, $b = 15$ and $c = 9$.

Heron's Area Formula Proof



note that $\cos C = \dfrac{\sqrt{b^2-h^2}}{b}$



By Law of Cosines,

$$c^2 = a^2+b^2-2ab\cos C$$

$$c^2 = a^2+b^2-2a\cancel{b}\left(\frac{\sqrt{b^2-h^2}}{\cancel{b}}\right)$$

Now try to get h by itself, since Area $= \frac{1}{2}ah$

$$c^2 = a^2+b^2-2a\sqrt{b^2-h^2}$$

$$2a\sqrt{b^2-h^2} = a^2+b^2-c^2$$

$$\sqrt{b^2-h^2} = \frac{a^2+b^2-c^2}{2a}$$

$$b^2-h^2 = \left(\frac{a^2+b^2-c^2}{2a}\right)^2$$

$$h^2 = b^2-\left(\frac{a^2+b^2-c^2}{2a}\right)^2$$

$$h^2 = \frac{4a^2b^2-(a^2+b^2-c^2)^2}{4a^2}$$

$$h = \frac{\sqrt{4a^2b^2-(a^2+b^2-c^2)^2}}{2a}$$

(notice difference of squares)

$$h = \frac{\sqrt{[2ab-(a^2+b^2-c^2)][2ab+(a^2+b^2-c^2)]}}{2a}$$

$$h = \frac{\sqrt{[-(a^2-2ab+b^2)+c^2][(a^2+2ab+b^2)-c^2]}}{2a}$$

$$h = \frac{\sqrt{[c^2-(a-b)^2][(a+b)^2-c^2]}}{2a}$$

$$h = \frac{\sqrt{(c-a+b)(c+a-b)(a+b-c)(a+b+c)}}{2a}$$

Let $t = a+b+c \Rightarrow h = \dfrac{\sqrt{(t-2a)(t-2b)(t-2c)(t)}}{2a}$

then

$c-a+b = a+b+c-2a$
$= t-2a$

$c+a-b = a+b+c-2b$
$= t-2b$

$a+b-c = a+b+c-2c$
$= t-2c$

and $\text{Area} = \frac{1}{2}ah$

$$\Rightarrow \text{Area} = \frac{1}{2}\cancel{a}\left(\frac{\sqrt{(t-2a)(t-2b)(t-2c)t}}{2\cancel{a}}\right)$$

$$= \frac{1}{4}\sqrt{(2s-2a)(2s-2b)(2s-2c)2s}$$

$$= \frac{1}{4}\sqrt{2^4(s-a)(s-b)(s-c)s}$$

$$= \frac{4}{4}\sqrt{(s-a)(s-b)(s-c)s}$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

w/ $s = \dfrac{a+b+c}{2}$ //

If $s = \dfrac{t}{2}$ then $t = 2s$

$\Rightarrow t-2a = 2s-2a$

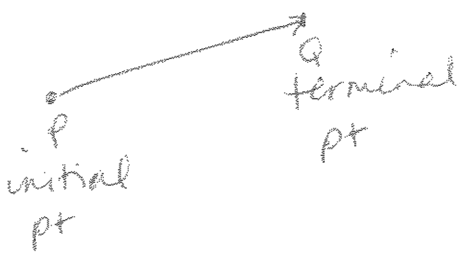
and $t-2b = 2s-2b$

$t-2c = 2s-2c$

64B

6.3 Vectors in a Plane

A __vector__ is basically a directed line segment. It has ① direction and ② magnitude (length). location does __not__ matter.



vector notation: $\overrightarrow{PQ}$

$\vec{v} = \overrightarrow{PQ}$ (in book, $\vec{v}$ will be **v**, i.e. boldfaced v)

magnitude (length)

$$\|\overrightarrow{PQ}\| = \|\vec{v}\|$$

__Component Form of Vector__

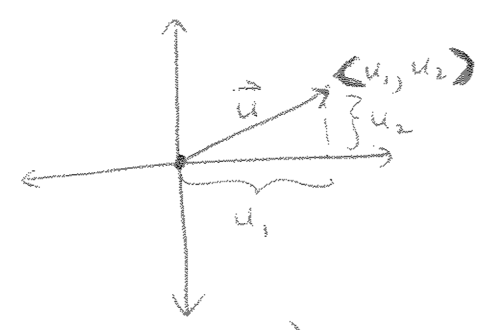
① For a vector $\vec{v}$ w/ initial pt P and terminal pt Q, $P(p_1, p_2)$ and $Q(q_1, q_2)$, then

$$\vec{v} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle$$

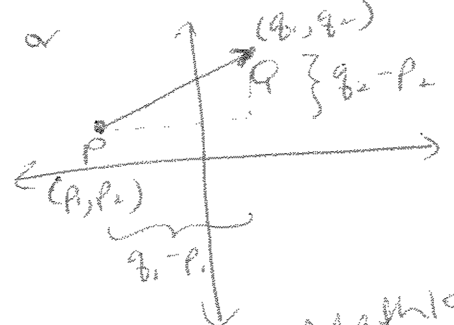
② The magnitude of $\overrightarrow{PQ} = \vec{v}$ is

$$\|\vec{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}$$

Since location doesn't matter, let's put it on coord. axes w/ initial pt at origin.



or



__Unit vector__

If $\|\vec{v}\| = 1$, then $\vec{v}$ is a unit vector, usually notated as $\hat{v}$.

6.3 (cont)

- $\|\vec{v}\| = 0 \iff \vec{v} = \vec{0}$.
- $\vec{u} = \vec{v} \iff u_1 = v_1$ and $u_2 = v_2$
- We can create a unit vector for $\vec{v}$, as

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}.$$

Vector Addition + Scalar multiplication

$\vec{u} = \langle u_1, u_2 \rangle$ $\quad \vec{v} = \langle v_1, v_2 \rangle$ $\quad k \in \mathbb{R}$

$\Rightarrow$ (1) $\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$

and (2) $k\vec{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle$

Scalar means a real # (i.e. a constant)

Properties of Vector Addition + Subtraction

(1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(2) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(3) $\vec{u} + \vec{0} = \vec{u}$

(4) $\vec{u} + (-\vec{u}) = \vec{0}$

(5) $c(d\vec{u}) = (cd)\vec{u}$

(6) $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

(7) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

(8) $1(\vec{u}) = \vec{u} = (\vec{u})1$ + $0(\vec{u}) = \vec{0}$

(9) $\|c\vec{v}\| = |c|\,\|\vec{v}\|$

6.3 (cont)

Ex 1 (a) Find component form of vector w/ initial pt (1,5) and terminal pt (−2,3).

(b) Find the unit vector.

Ex 2 Find $\vec{u}+\vec{v}$ if $\vec{u} = \langle -1, 4 \rangle$ + $\vec{v} = \langle 2, 3 \rangle$.

Find $\vec{u} - 3\vec{v}$.

6.3 (cont)

Ex 3 Find the vector $\vec{v}$ w/ magnitude of 3 and in the direction of $\vec{u} = \langle -4, 5 \rangle$.

$\boxed{\begin{array}{l}\underline{\text{Standard unit vectors}} \\ \hat{i} = \langle 1, 0 \rangle \text{ and } \hat{j} = \langle 0, 1 \rangle.\end{array}}$

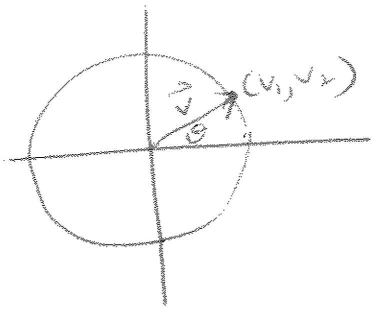
$\Rightarrow \vec{v} = \langle v_1, v_2 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle = v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle$

$\Leftrightarrow \vec{v} = v_1 \hat{i} + v_2 \hat{j}$

Ex 4 If $\vec{v} = 7\hat{i} - 2\hat{j}$ and $\vec{w} = -2\hat{i} + \hat{j}$, find $2\vec{v} + \vec{w}$.

6.3 (cont)

Direction Angles



$\vec{v} = \langle v_1, v_2 \rangle = \langle \|\vec{v}\| \cos\theta, \|\vec{v}\| \sin\theta \rangle$

$\vec{v} = \|\vec{v}\| \langle \cos\theta, \sin\theta \rangle$

$\dfrac{\vec{v}}{\|\vec{v}\|} = \hat{v} = \langle \cos\theta, \sin\theta \rangle$

θ = direction angle for our vector

So, $\dfrac{v_1}{\|\vec{v}\|} = \cos\theta \qquad \dfrac{v_2}{\|\vec{v}\|} = \sin\theta$

and $\tan\theta = \dfrac{v_2/\|\vec{v}\|}{v_1/\|\vec{v}\|} = \dfrac{v_2}{v_1}$.

<u>Ex 5</u> Find the direction angle of $\vec{v} = \langle -3, 6 \rangle$.

6.3 (cont)

Ex 6 Find $\vec{v}$ if $\theta = 30°$ & magnitude is 6.

6.4 Vectors and Dot Products

Dot Product (Algebraic Defn)

If $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$, then $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$

Properties of Dot Products

① $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

② $\vec{0} \cdot \vec{v} = 0$

③ $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

④ $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

⑤ $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$

$$\vec{v} \cdot \vec{v} = v_1 v_1 + v_2 v_2$$
$$= v_1^2 + v_2^2$$
$$= \left(\sqrt{v_1^2 + v_2^2}\right)^2$$
$$= \|\vec{v}\|^2$$

So we can use dot product, if we want, to get length.

Ex 1 Find dot product

(a) $\vec{u} = \langle 5, 12 \rangle$ $\quad \vec{v} = \langle -3, 2 \rangle$

(b) $\vec{u} = 3\hat{i} + 4\hat{j}$ $\quad \vec{v} = 7\hat{i} - 2\hat{j}$

6.4 (cont)

Geometric defn of Dot Product

$$\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta \quad \text{where } \theta = \text{angle between } \vec{u} \text{ and } \vec{v}.}$$

★ (see proof on pg 72A)

what if $\vec{u} \cdot \vec{v} = 0$?

$\Rightarrow \|\vec{u}\| \, \|\vec{v}\| \cos\theta = 0 \quad \Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \pi/2$ or $3\pi/2$

$\Rightarrow \vec{u} \perp \vec{v}$!

$$\boxed{\vec{u} \cdot \vec{v} = 0 \iff \vec{u} \perp \vec{v}}$$

Ex 2 Find the angle between $\vec{u} = -6\hat{\imath} - 3\hat{\jmath}$ and $\vec{v} = -8\hat{\imath} + 4\hat{\jmath}$.

If $\cos\theta = -1$, $\theta = \pi$ (opposite direction for $\vec{u}$ & $\vec{v}$)

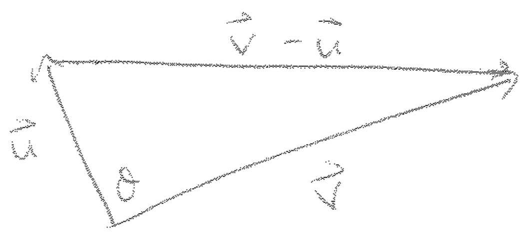
$\cos\theta = 0$, $\theta = \pm\pi/2$ ($\vec{u} \perp \vec{v}$)

$\cos\theta = 1$, $\theta = 0$ (same direction for $\vec{u}$ & $\vec{v}$)

$0 < \cos\theta < 1$, θ acute

$-1 < \cos\theta < 0$, θ obtuse

Proof of $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$.



Given vectors $\vec{u}$ and $\vec{v}$, form triangle as shown.

Then, use Law of Cosines.

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \|\vec{v}\| \cos\theta$$

$$\|\vec{v} - \vec{u}\|^2 - \|\vec{u}\|^2 - \|\vec{v}\|^2 = -2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

but $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u} = u_1^2 + u_2^2$

and $\vec{v} - \vec{u} = \langle v_1 - u_1, v_2 - u_2 \rangle$

$$(v_1 - u_1)^2 + (v_2 - u_2)^2 - (u_1^2 + u_2^2) - (v_1^2 + v_2^2) = -2\|\vec{u}\| \|\vec{v}\| \cos\theta$$

$$\cancel{v_1^2} - 2u_1v_1 + \cancel{u_1^2} + \cancel{v_2^2} - 2v_2u_2 + \cancel{u_2^2} - \cancel{u_1^2} - \cancel{u_2^2} - \cancel{v_1^2} - \cancel{v_2^2} = -2\|\vec{u}\| \|\vec{v}\| \cos\theta$$

$$-2(u_1v_1 + v_2u_2) = -2\|\vec{u}\| \|\vec{v}\| \cos\theta$$

$$u_1v_1 + u_2v_2 = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta \ /\!/$$

6.4 (cont)

Ex 3 Decide if $\vec{u}$ & $\vec{v}$ are $\parallel$, $\perp$ or neither.

(a) $\vec{u} = \langle 1, 0 \rangle$ $\quad \vec{v} = \langle -2, 2 \rangle$

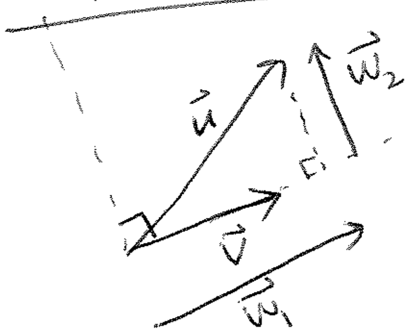
(b) $\vec{u} = \langle 3, 5 \rangle$ $\quad \vec{v} = \langle -6, -10 \rangle$

(c) $\vec{u} = \langle -1, 2 \rangle$ $\quad \vec{v} = \langle 6, 3 \rangle$

Let's say we want to define $\vec{u}$ as $\vec{u} = \vec{w}_1 + \vec{w}_2 \ni \vec{w}_1$ is $\parallel$ to some vector $\vec{v}$, and $\vec{w}_2$ is $\perp$ to $\vec{v}$, i.e. we want to write $\vec{u}$ as the sum of component vectors $\vec{w}_1 + \vec{w}_2$.

We need the piece of $\vec{u}$ that is in the direction of $\vec{v}$ + then the piece of $\vec{u}$ that's $\perp$ to that direction.

6.4 (cont)



$\vec{u} = \vec{w}_1 + \vec{w}_2$

We say "$\vec{w}_1$ is the projection of $\vec{u}$ onto $\vec{v}$" and write it as

$$\vec{w}_1 = \text{proj}_v \vec{u}$$

We know $\vec{w}_1 = c\vec{v}$ for some $c \in \mathbb{R}$.

$$\Rightarrow \vec{u} = c\vec{v} + \vec{w}_2$$

$$\Rightarrow \vec{u} \cdot \vec{v} = (c\vec{v} + \vec{w}_2) \cdot \vec{v}$$
$$= c(\vec{v} \cdot \vec{v}) + (\vec{w}_2 \cdot \vec{v})$$

but $\vec{w}_2 \perp \vec{v}$
$\Leftrightarrow \vec{w}_2 \cdot \vec{v} = 0$

$$\Rightarrow \vec{u} \cdot \vec{v} = c\|\vec{v}\|^2$$

$$\Rightarrow c = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}$$

$$\Rightarrow \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)\vec{v} + \vec{w}_2$$

$$\Rightarrow \boxed{\vec{w}_1 = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)\vec{v} = \text{proj}_v \vec{u}}$$

and $\vec{w}_2 = \vec{u} - \vec{w}_1$

6.4 (cont)

Ex 4 Find $\text{proj}_{\vec{v}}\vec{u}$.

(a) $\vec{u} = \langle 4, 2 \rangle$ $\quad \vec{v} = \langle 1, -2 \rangle$

(b) $\vec{u} = \langle 2, -1 \rangle$ $\quad \vec{v} = \langle 1, 3 \rangle$