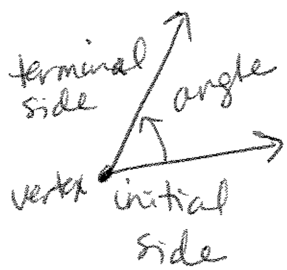
4.1 Radian + Degree Measure

Vocab

trigonometry ⇒ "measurement of triangles"

angle ⇒ intersection of 2 rays



standard position ⇒ an angle in standard position on a Cartesian coordinate system is oriented so its vertex is at the origin, its initial side is on the positive x-axis.

positive angles ⇒ measured from standard position in a counter-clockwise fashion

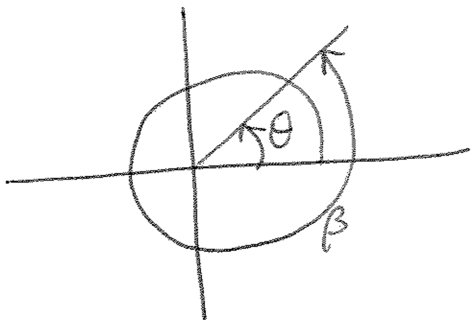
negative angles ⇒ measured clockwise

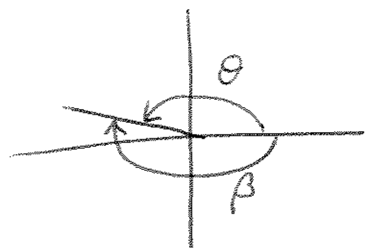
coterminal angles ⇒ angles that have same initial + terminal sides (i.e. every angle can be labelled infinitely many ways)

central angle ⇒ an angle whose vertex is the center of a circle

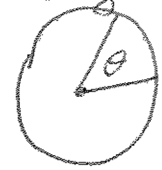
4.1 (cont)

θ and β are coterminal





central angle



Radian Defn

One radian is the measure of a central angle θ that intercepts an arc s equal in length to r (the radius of the circle).

i.e. $\theta = \frac{s}{r}$ (unitless measure)

circumference of a circle $C = 2\pi r$

$\Rightarrow \frac{C}{r} = 2\pi$ i.e. angle of full circle is 2π radians

$\Leftrightarrow 360° = 2\pi \Leftrightarrow \boxed{\pi = 180°}$

$1'$ = one minute $= \frac{1}{60}(1°)$

$1''$ = one second $= \frac{1}{60}(1') = \frac{1}{3600}(1°)$

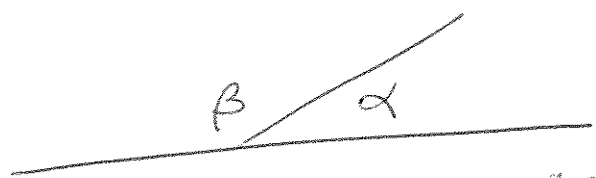
QII $\pi/2 < \theta < \pi$	QI $0 \le \theta < \pi/2$
QIII $\pi < \theta < \frac{3\pi}{2}$	QIV $\frac{3\pi}{2} < \theta < 2\pi$

4.1 (cont)

<u>complementary angles</u> $\Rightarrow$ α & β are complementary if $\alpha + \beta = \pi/2$ (or $90°$)

<u>supplementary angles</u> $\Rightarrow$ α & β are supplementary if $\alpha + \beta = \pi$ (or $180°$)

complementary angles



supplementary angles

Ex 1 Which quadrant are these angles in? Sketch them.

(a) -1

(b) $3\pi/8$

(c) $5\pi/3$

(d) $7\pi/4$

(e) 3

4.1 (cont)

Ex 2 Find 2 coterminal angles.

(a) $\frac{-11\pi}{6}$

(b) $3\pi/4$

Ex 3 Rewrite each angle in radians.

(a) $-270°$

(b) $144°$

Ex 4 Rewrite each angle in degrees.

(a) $\frac{13\pi}{2}$

(b) $\frac{7\pi}{8}$

4.1 (cont)

Arc Length

For a circle of radius r, a central angle θ intercepts an arc of length s $\ni$

$$s = r\theta \qquad (\theta \text{ measured in radians})$$

Ex 5 find arc length on circle of radius 9 ft. intercepted by $\theta = 60°$.

Linear + Angular Speeds

$$v = \frac{\text{arc length}}{\text{time}} = \frac{s}{t} \qquad (\text{linear speed})$$

$$\omega = \frac{\text{central angle}}{\text{time}} = \frac{\theta}{t} \qquad (\text{angular speed})$$

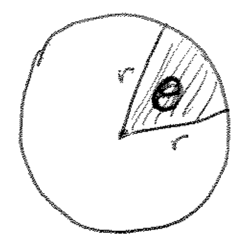
(θ measured in radians)

4.1 (cont)

Ex 6 A carousel w/ a 50 ft diameter makes 4 revolutions per minute.

(a) find the angular speed.

(b) find the linear speed.

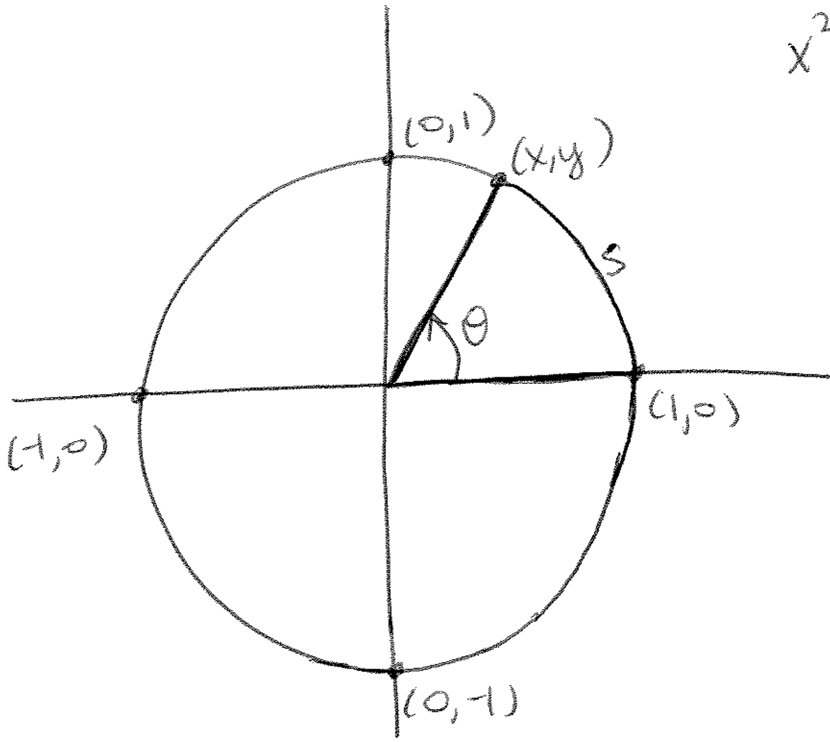


Area of a sector of a circle (aka a "pie" piece)

$$A = \frac{1}{2} r^2 \theta \qquad (\theta \text{ in radians})$$

Ex 7 Find the area of the sector of a circle w/ radius = 3 in + $\theta = \pi/4$.

4.2 Trig Fns: The Unit Circle



$x^2+y^2=1$ **unit circle**

(circle centered at origin, radius = 1)

$s=\theta$
(since $r=1$)

"Wrap" a real # line around unit circle.
For any bit of arc length travelled around the circle, we land on a pt (x,y) on the circle.

$x \in [0,1]$ & $y \in [0,1]$

$$\boxed{(x,y) = (\cos\theta, \sin\theta)}$$
on unit circle

(i.e. we can get to any pt on circle by going around θ radians)

4.2 (cont)

Defn of Trig Fns

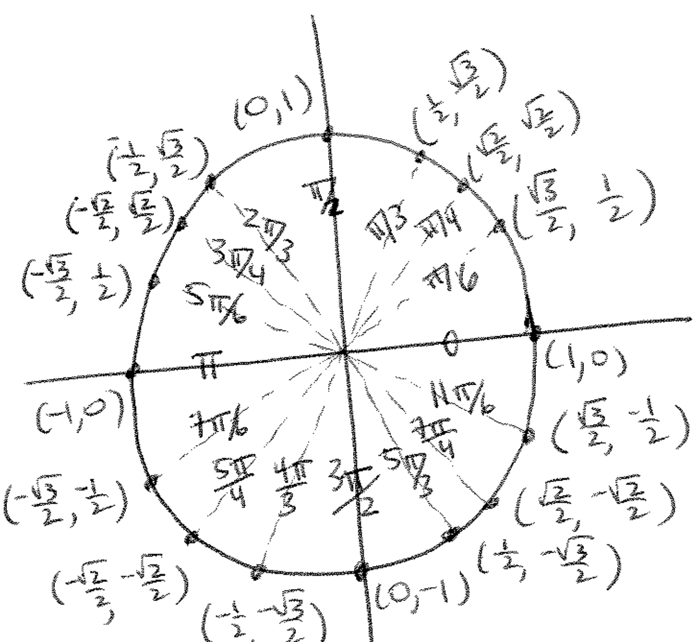
Let $\theta \in \mathbb{R}$ & (x,y) is point on unit circle corresponding to θ. Then,

$x = \cos\theta$ $\qquad \frac{y}{x} = \tan\theta \; (x \neq 0)$ $\qquad \frac{1}{y} = \csc\theta \; (y \neq 0)$

$y = \sin\theta$ $\qquad \frac{x}{y} = \cot\theta \; (y \neq 0)$ $\qquad \frac{1}{x} = \sec\theta \; (x \neq 0)$

Typical Angles

☆ memorize this



$\Rightarrow \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$

$\sin\left(7\pi/4\right) = \frac{-\sqrt{2}}{2}$

$\cos\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2}$

etc.

Notice along line $y = x$ (it cuts QI in half, i.e. $\theta = \pi/4$) $\Rightarrow$ $x^2 + x^2 = 1 \Leftrightarrow 2x^2 = 1 \Leftrightarrow x^2 = 1/2$

$\Rightarrow x = \pm 1/\sqrt{2} = \pm\sqrt{2}/2 = y.$

4.2 (cont)

Ex1 Evaluate all six trig fns of the following angles.

(a) $\theta = 5\pi/2$

(b) $\theta = 2\pi/3$

4.2 (cont)

Domain & Range of Trig Fns

<u>domain</u> ⇒ set of allowable inputs

<u>range</u> ⇒ set of outputs

Consider
$y = \cos\theta$
(or $y = \sin\theta$)

$-1 \le \sin\theta \le 1 \qquad -1 \le \cos\theta \le 1$

⇒ range is $[-1, 1]$
but θ could be anything, i.e. domain is $\theta \in \mathbb{R}$.

<u>Notice</u> $\sin(\theta + 2\pi) = \sin(\theta + 4\pi) = \sin(\theta - 2\pi)$
$= \sin\theta$
$\ldots = \sin(\theta + 2n\pi) \quad \forall n \in \mathbb{Z}$

(Also true for cosine.)

<u>Periodic Fn</u>

A fn f is periodic if $\exists$ positive $\mathbb{R}$ # $c \ni$
$f(t+c) = f(t) \quad \forall\, t \in$ domain of f.

c is called the <u>period</u>

trig fns are periodic!!!

4.2 (cont)

Even + Odd Trig Fns

Even $\cos(\theta) = \cos(-\theta)$

$\sec(\theta) = \sec(-\theta)$

Odd $\sin(-\theta) = -\sin\theta$, $\csc(-\theta) = -\csc\theta$,

$\tan(-\theta) = -\tan\theta$, $\cot(-\theta) = -\cot\theta$

Ex 2 If $\sin(-\theta) = 3/8$, evaluate the following.

(a) $\sin\theta$

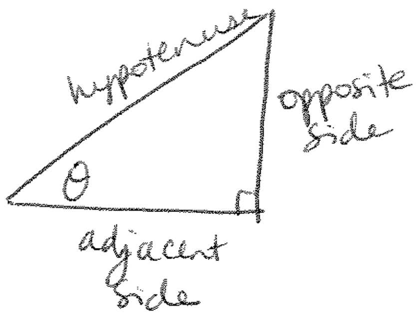
(b) $\csc\theta$

Ex 3 If $\cos\theta = 4/5$, evaluate the following.

(a) $\cos(\pi - \theta)$

(b) $\cos(\pi + \theta)$

4.3 Right Triangle Trigonometry



Defn of Trig Fns

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

Fundamental Trig Identities

① $\sin\theta = \dfrac{1}{\csc\theta}$ $\qquad \cos\theta = \dfrac{1}{\sec\theta}$ $\qquad \tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\cot\theta}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$ $\qquad \sec\theta = \dfrac{1}{\cos\theta}$ $\qquad \csc\theta = \dfrac{1}{\sin\theta}$

② Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Prove $\sin^2\theta + \cos^2\theta = 1$.

pf From right Δ, we know $(\text{adj})^2 + (\text{opp})^2 = (\text{hyp})^2$

Divide both sides by $(\text{hyp})^2 \Rightarrow \left(\dfrac{\text{adj}}{\text{hyp}}\right)^2 + \left(\dfrac{\text{opp}}{\text{hyp}}\right)^2 = 1$

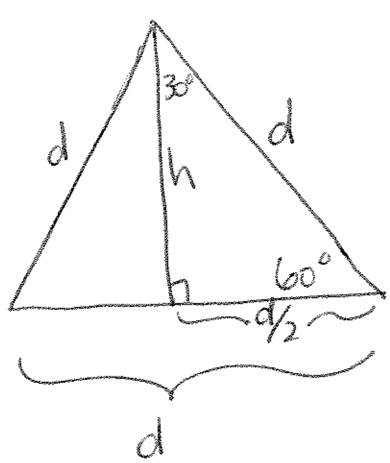
$$\Rightarrow \cos^2\theta + \sin^2\theta = 1 \quad //$$

4.3 (cont)

Ex 1 Sketch a right triangle corresponding to $\cos\theta = 5/7$. Find all other trig fns of θ.

Prove $\sin 30° = 1/2$ + $\cos 30° = \sqrt{3}/2$.

Pf Start w/ equilateral Δ.



By Pythagorean Thm,

$$h^2 + \left(\frac{d}{2}\right)^2 = d^2$$

$$h^2 + \frac{d^2}{4} = d^2 \Rightarrow h^2 = d^2 - \frac{d^2}{4}$$

$$\Rightarrow h^2 = \frac{3}{4}d^2 \Rightarrow h = \frac{\sqrt{3}}{2}d$$

$$\Rightarrow \sin 30° = \frac{opp}{hyp} = \frac{d/2}{d} = \frac{1}{2}$$

$$\cos 30° = \frac{adj}{hyp} = \frac{h}{d} = \frac{\sqrt{3}/2\, d}{d} = \frac{\sqrt{3}}{2}.$$

:)

4.3 (cont)

Ex 2 Sketch a right triangle corresponding to $\csc\theta = \frac{17}{4}$. Use the Pythagorean Thm to find the third side, + find all other trig fns of θ.

Ex 3 Use $\tan\beta = 5$ to find

(a) $\cos\beta$ (b) $\csc\beta$ (c) $\tan(90^\circ - \beta)$

4.3 (cont)

Ex 4 Use trig identities to verify

(a) $\sin^2\theta - \cos^2\theta = 2\sin^2\theta - 1$

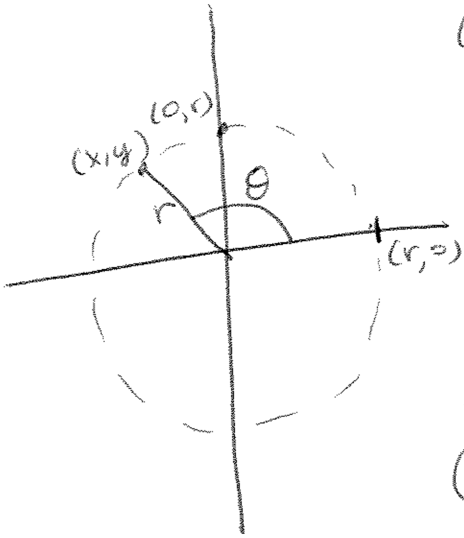
(b) $\dfrac{\tan\beta + \cot\beta}{\tan\beta} = \csc^2\beta$

4.3 (cont)

Ex 5 A 6-ft person walks from the base of a tower directly toward the tip of the shadow cast by the tower. When the person is 132 ft from the tower + 3 feet from the tip of the shadow, the person's shadow starts to appear beyond the tower's shadow. What is the height of the tower?

Ex 6 You are hiking down a mountain w/ vertical height of 1100 ft. The distance from the top of the mountain to the base is 2200 ft. What is the angle of elevation?

4.4 Trigonometric Fns of Any Angle



(x,y) is generic pt on circle centered at origin w/ radius r

(i.e. r = distance from origin to (x,y))

circle: $x^2+y^2=r^2$

and $(\cos^2\theta+\sin^2\theta)r^2=r^2$ [since $\cos^2\theta+\sin^2\theta=1$]

$(r\cos\theta)^2+(r\sin\theta)^2=r^2$

i.e. we can say $x=r\cos\theta$ + $y=r\sin\theta$

$\Rightarrow \cos\theta=\frac{x}{r}$ $\sin\theta=\frac{y}{r}$

(not just acute angles)

Defn of Trig Fns of Any Angle to (x,y) on terminal side.

θ = angle in standard position

$r=\sqrt{x^2+y^2}\neq 0$

$\Rightarrow \sin\theta=\frac{y}{r}$ $\cos\theta=\frac{x}{r}$ $\sec\theta=\frac{r}{x}$ $(x\neq 0)$

$\tan\theta=\frac{y}{x}$ $(x\neq 0)$ $\cot\theta=\frac{x}{y}$ $(y\neq 0)$ $\csc\theta=\frac{r}{y}$ $(y\neq 0)$

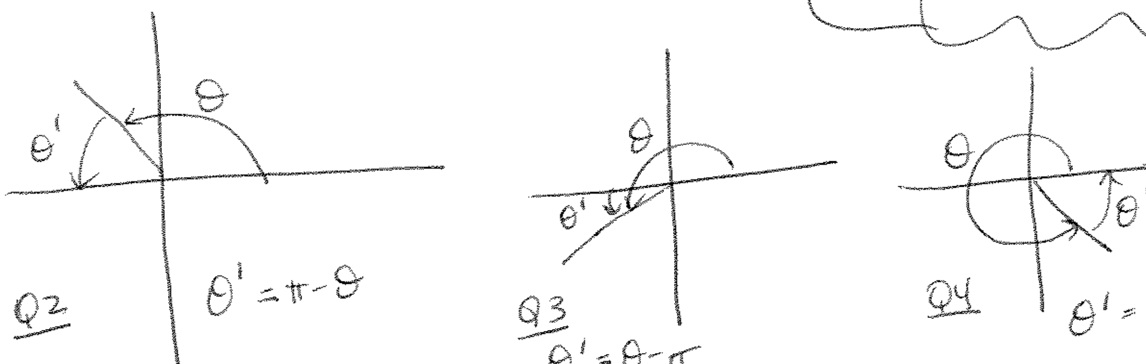
Ex 1 Let (4, -3) be a pt on terminal side of θ. Find $\sin\theta$, $\cos\theta$ + $\tan\theta$.

4.4 (cont)

Defn of Reference Angle

Let θ be an angle in standard position. Its reference angle is the acute angle θ' formed by the terminal side of θ & the horizontal axis.

QII	QI
$\pi/2 < \theta < \pi$ $\sin\theta > 0$ $\cos\theta < 0$ $\tan\theta < 0$	$0 < \theta < \pi/2$ $\sin\theta > 0$ $\cos\theta > 0$ $\tan\theta > 0$
$\pi < \theta < 3\pi/2$ $\sin\theta < 0$ $\cos\theta < 0$ $\tan\theta > 0$	$\frac{3\pi}{2} < \theta < 2\pi$ $\sin\theta < 0$ $\cos\theta > 0$ $\tan\theta < 0$
QIII	QIV



Q2 $\theta' = \pi - \theta$

Q3 $\theta' = \theta - \pi$

Q4 $\theta' = 2\pi - \theta$

Ex2 Find reference angle θ'.

(a) $\theta = -120°$

(b) $\theta = \frac{11\pi}{6}$

(c) $\theta = 2\pi/3$

4.4 (cont)

Ex 3 Determine exact values of the six trig fns of angle in standard position whose terminal side contains the point $(-3, -7)$.

Ex 4 Terminal side of θ lies on line $4x + 3y = 0$ in Q4. Find values of 6 trig fns.

4.4 (cont)

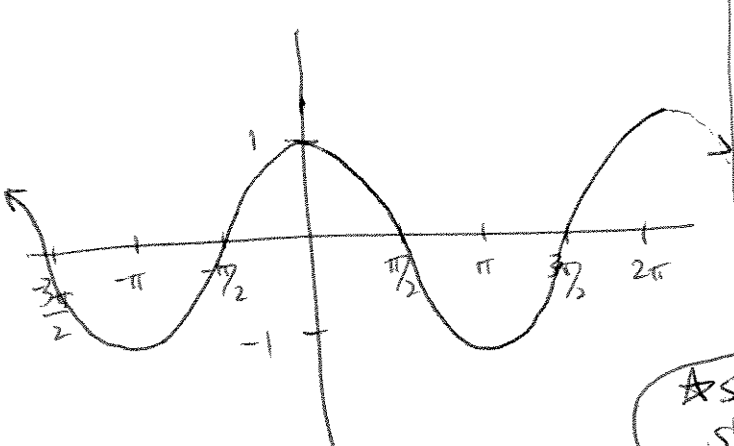
Ex 5 Find the values of the six trig fns of θ given $\csc\theta = 4$ + $\cot\theta < 0$.

4.5 Graphs of Sine + Cosine Fns

Think of sine + cosine as functions. We could use $f(\theta) = \cos\theta$ or $g(\theta) = \sin\theta$. Typically, we just go back to our favorite variables, x and y.

$y = \cos x$ or $y = \sin x$.

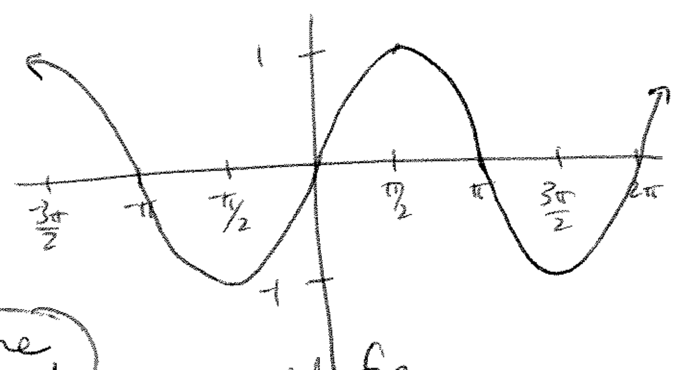
$\boxed{y = \cos x}$



even fn
<u>range</u>: $-1 \le y \le 1$
<u>domain</u>: $x \in \mathbb{R}$

same shape!

$\boxed{y = \sin x}$



odd fn
<u>range</u>: $-1 \le y \le 1$
<u>domain</u>: $x \in \mathbb{R}$

<u>period</u> $= 2\pi$
(i.e. it repeats itself every 2π units)

Let's look now at

$y = a\sin(bx - c) + d$ + remember how a, b, c, d affect graph.

horizontal shift:
vertical shift:
horizontal stretch:
vertical stretch:

4.5 (cont)

<u>Amplitude</u> ⇒ For $y = a\sin(bx-c)+d$ or
$y = a\cos(bx-c)+d$, amplitude is $|a|$.

<u>Period</u> ⇒ For $y=a\sin(bx-c)+d$ or $y=a\cos(bx-c)+d$
period is $\frac{2\pi}{b}$. $(b>0)$

<u>Ex 1</u> Compared to $y=\sin x$, what happens for these graphs? Give amplitude + period.

(a) $y = 3\sin\left(\frac{2}{3}x\right)$

(b) $y = -2\sin(2x)+3$

(c) $y = \sin(4x+1)-2$

4.5 (cont)

Ex 2 Sketch graph of these functions.

(a) $y = 4\cos x$

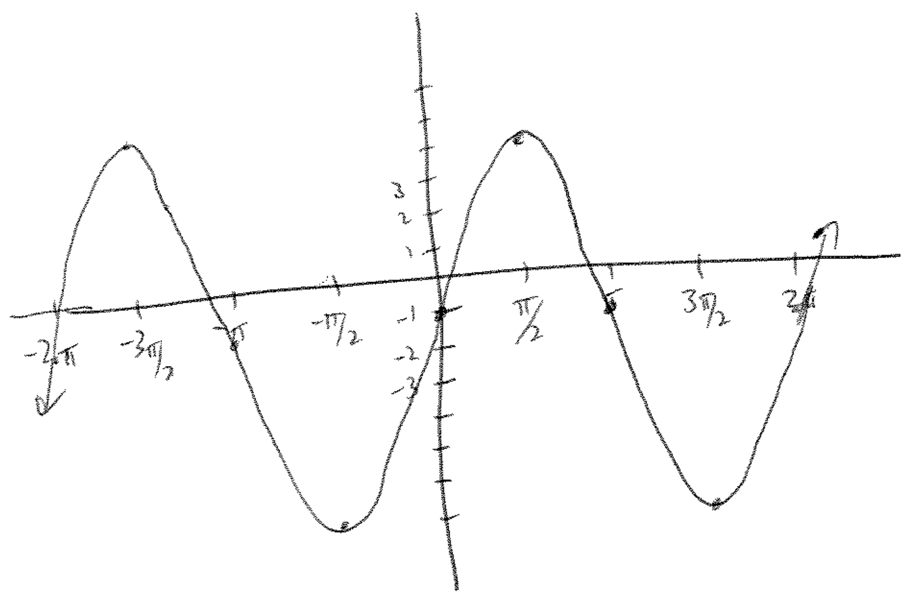
(b) $y = -2\sin\left(\frac{\pi x}{4}\right)$

(c) $y = 4\cos\left(x + \frac{\pi}{4}\right) - 3$

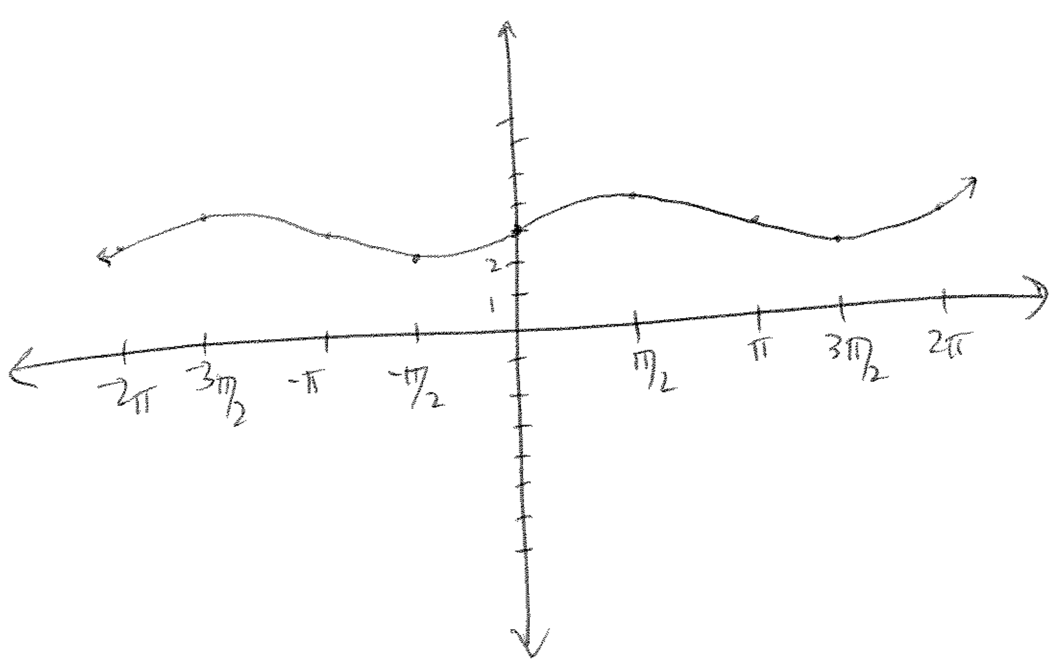
4.5 (cont)

Ex 3 Find a + d ∋ the graph of f matches these images, for $f(x) = a \sin x + d$

(a)



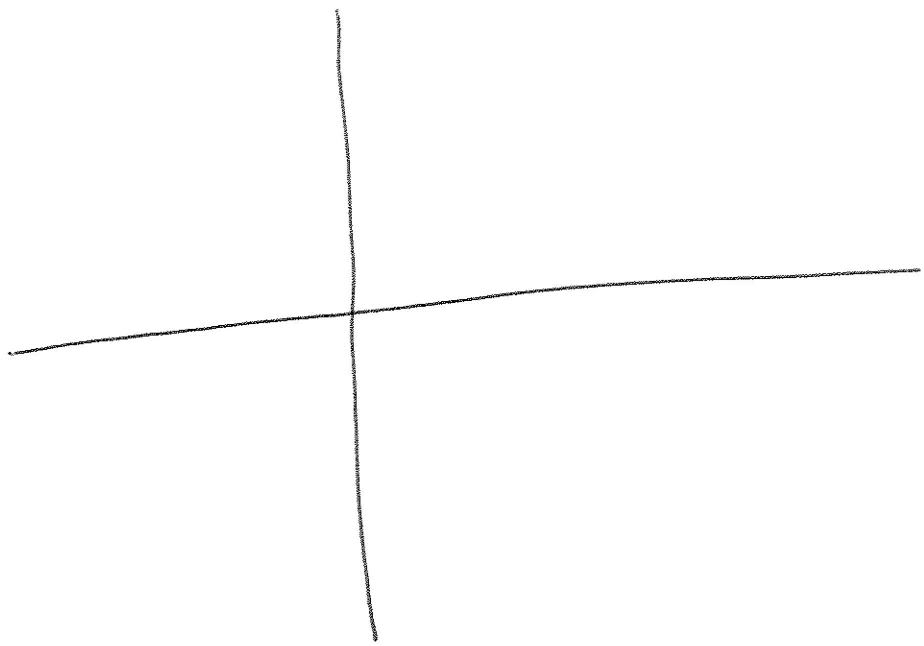
(b)



4.6 Graph of the Tangent Function

Look at graph of $y = \tan x$.

x	y
0	
$\pi/4$	
$-\pi/4$	
$\pi/2$	
$-\pi/2$	
π	
$3\pi/4$	
$5\pi/4$	



period = ______

amplitude?

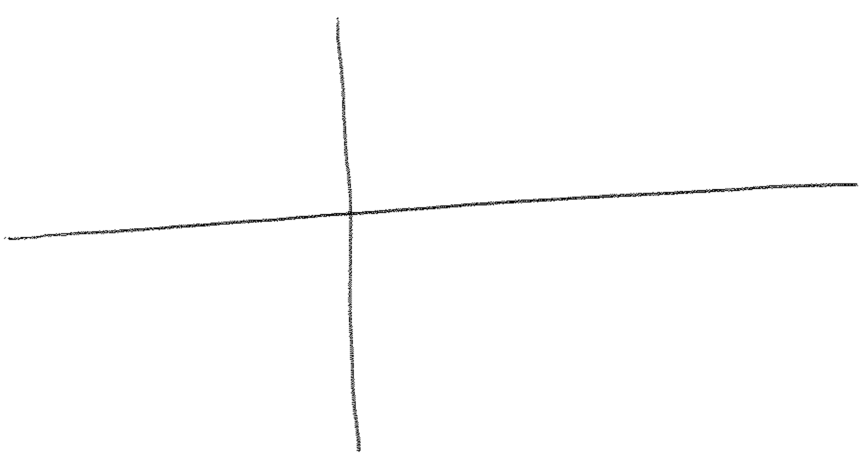
VA:

domain:

range:

What does $y = a\tan(bx - c) + d$ do to original fn?

graph $y = \cot x$



period:

amplitude:

VA:

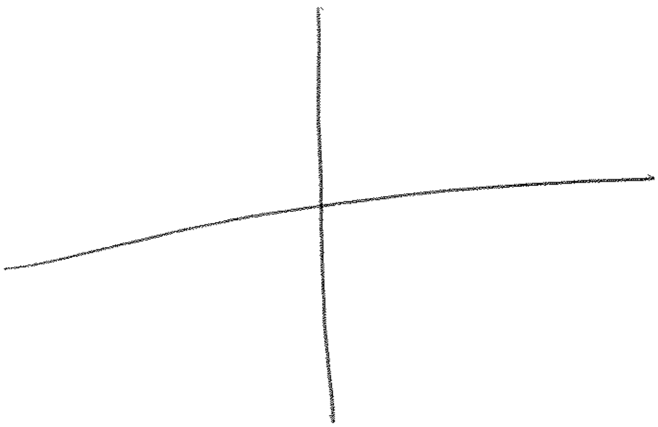
domain:

range:

4.6 (cont)

symmetry?

$y = \csc x$



domain:
range:

VA:
period:

$y = \sec x$

domain:
range:

VA:
period:

Ex 1 Sketch graph of these fns.

(a) $y = -3\tan(\pi x)$

(b) $y = \csc\left(\frac{x}{3}\right)$

4.6 (cont)

Ex 2 Sketch graph of these fns.

(a) $y = 2\cot(x + \pi/2)$

(b) $y = 3\sec(2x - \pi)$

Ex 3 Use graph to solve $\cot x = 1$ on $[-2\pi, 2\pi]$.

4.6 (cont)

☆ all trig fn graphs on pg 338

Ex 4 Graph damped fn

$y = x \sin x.$

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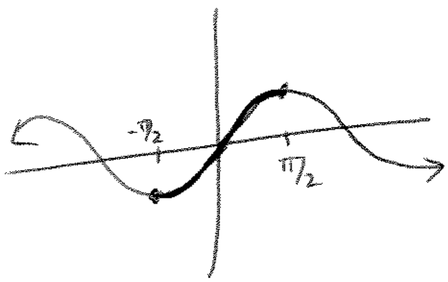
4.7 Inverse Trig Fns

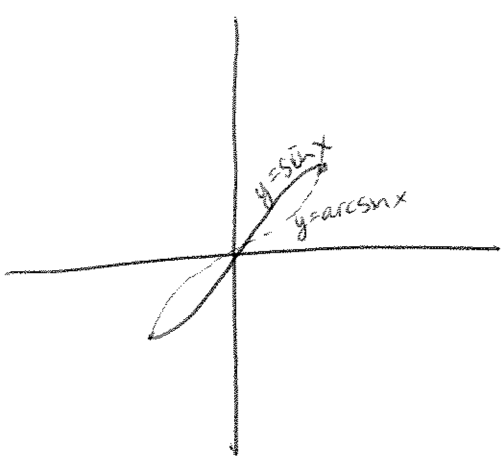
Remember fns only have an inverse if the graph passes the horizontal line test.

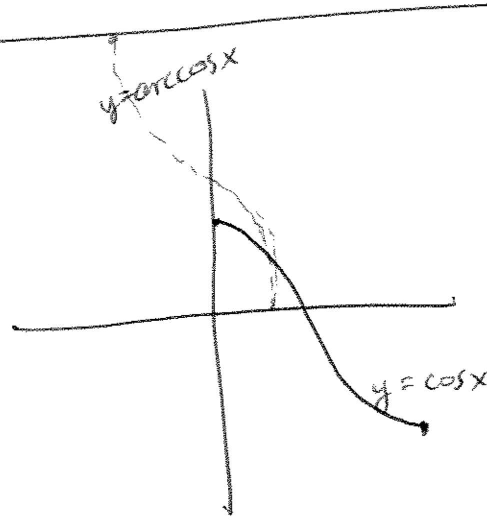


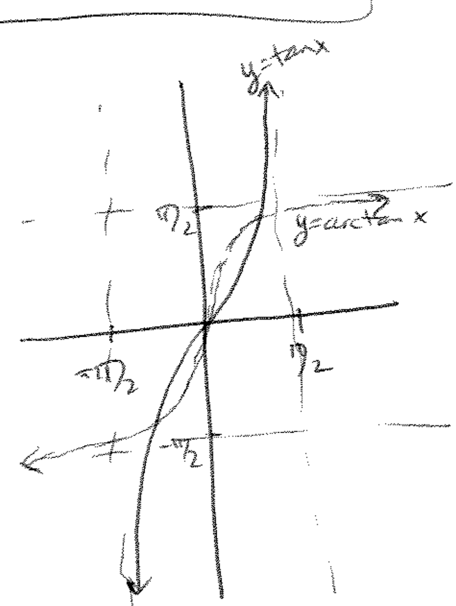
We ~~want~~ to find an inverse fn ~~for~~ $y = \sin x$, so we restrict the domain to $x \in [-\pi/2, \pi/2]$.

Inverse Functions

$y = \arcsin x$ $\Leftrightarrow$	$\sin y = x$,	$x \in [-1, 1]$,	$y \in [-\pi/2, \pi/2]$
$y = \arccos x$ $\Leftrightarrow$	$\cos y = x$,	$x \in [-1, 1]$,	$y \in [0, \pi]$
$y = \arctan x$ $\Leftrightarrow$	$\tan y = x$,	$x \in (-\infty, \infty)$,	$y \in (-\pi/2, \pi/2)$







4.7 (cont)

Ex 1 Evaluate (w/o a calculator).

(a) $\arccos 0 =$

(b) $\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right) =$

(c) $\cos^{-1}(1) =$

(d) $\arctan(\sqrt{3}) =$

(e) $\arcsin(0) =$

4.7 (cont)

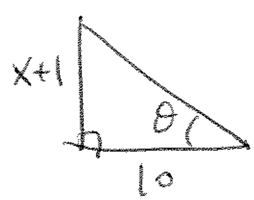
Ex 2 Use a calculator to evaluate.

(a) arctan 2.8

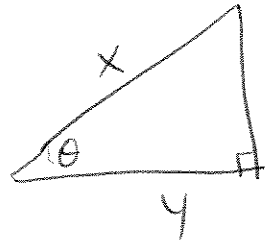
(b) $\cos^{-1}(0.26)$

Ex 3 Use an universe trig fn to write θ as function of x.

(a)



(b)



4.7 (cont)

Ex 4 Evaluate.

(a) $\arccos\left(\cos \frac{7\pi}{2}\right)$

(b) $\sin[\arcsin(-0.2)]$

(c) $\tan\left[\arcsin\left(-\frac{3}{4}\right)\right]$

(d) $\sec\left(\arcsin \frac{4}{5}\right)$

4.7 (cont)

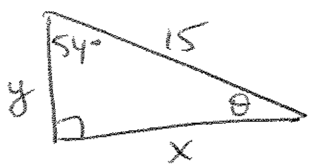
Ex 5 Transform these trig expressions into algebraic expressions.

(a) $\sec(\arctan(3x))$

(b) $\cos(\arcsin(x-1))$

4.8 Trig Applications and Models

Ex 1 Solve the right Δ.

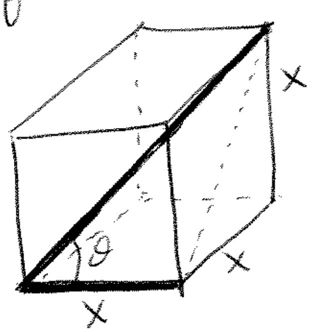


Ex 2 Find the altitude of the isosceles triangle where the base angle is 18° + the base leg length is 10 meters.

4.8 (cont)

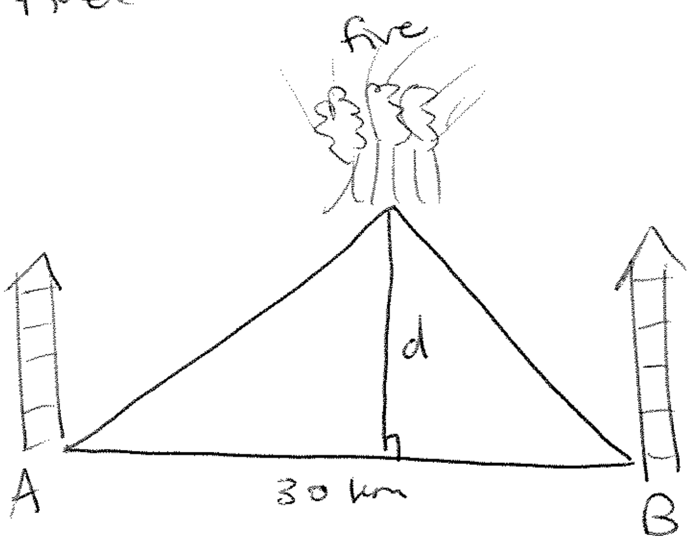
Ex 3 The length of a shadow of a tree is 125 ft when the angle of elevation of the sun is 33°. Approximate the height of the tree.

Ex 4 Determine the angle between the diagonal of a cube and its edge.



4.8 (cont)

Ex 5 Two fire towers are 30 km apart, where tower A is due west of tower B. A fire is spotted from the towers, and the bearings from A + B are E14°N + W34°N, respectively. Find the distance d of the fire from $\overline{AB}$.



4.8 (cont)

Ex 6 While traveling across flat land, you notice a mountain directly in front of you. The angle of elevation to the peak is $2.5°$. After you drive 17 miles closer to the mountain, the angle of elevation is $9°$. Approximate the height of the mountain.