A couple of you have asked if you can find the average rate of change in 2 ways:

1. finding the slope of the line between the 2 pts (the way taught in class in beginning of 2.3).

and 2. finding the average of the derivatives at each of the 2 pts.

Let's look at some examples.

(A) Let $f(x) = ax^2 + bx + c$ $\quad a, b, c \in \mathbb{R}$
(i.e. any quadratic function)

The avg rate of change from x_1 to x_2 is given by

1. $$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(ax_2^2 + bx_2 + c) - (ax_1^2 + bx_1 + c)}{x_2 - x_1}$$

$$= \frac{a(x_2^2 - x_1^2) + b(x_2 - x_1) + (c - c)}{x_2 - x_1}$$

$$= \frac{a(x_2 - x_1)(x_1 + x_2) + b(x_2 - x_1)}{(x_2 - x_1)}$$

$$= \frac{\cancel{(x_2 - x_1)}\left[a(x_1+x_2)+b\right]}{\cancel{(x_2-x_1)}}$$

$$\boxed{m = a(x_1+x_2)+b}$$

Now, let's calculate idea 2

$$\frac{f'(x_1)+f'(x_2)}{2} \quad \text{(avg of derivatives)}$$

we know $f'(x) = 2ax + b$

$$\Rightarrow \frac{f'(x_1)+f'(x_2)}{2} = \frac{2ax_1 + b + 2ax_2 + b}{2}$$

$$= \frac{2a(x_1+x_2) + 2b}{2}$$

$$= \frac{\cancel{2}\left[a(x_1+x_2)+b\right]}{\cancel{2}}$$

$$\boxed{\frac{f'(x_1)+f'(x_2)}{2} = a(x_1+x_2)+b}$$

$\Rightarrow$ idea 1 and idea 2 yield the same result! But, so far, this is only true for quadratic functions.

(B) Let's now look at the same work

for $g(x) = ax^3 + bx^2 + cx + d$ (a cubic polynomial)

① $m = \dfrac{g(x_2) - g(x_1)}{x_2 - x_1} = \dfrac{(ax_2^3 + bx_2^2 + cx_2 + d) - (ax_1^3 + bx_1^2 + cx_1 + d)}{x_2 - x_1}$

$= \dfrac{a(x_2^3 - x_1^3) + b(x_2^2 - x_1^2) + c(x_2 - x_1) + (d - d)}{(x_2 - x_1)}$

$= \dfrac{a(x_2 - x_1)(x_2^2 + x_1x_2 + x_1^2) + b(x_2 - x_1)(x_2 + x_1) + c(x_2 - x_1)}{(x_2 - x_1)}$

$= \dfrac{\cancel{(x_2 - x_1)}[a(x_2^2 + x_1x_2 + x_1^2) + b(x_2 + x_1) + c]}{\cancel{(x_2 - x_1)}}$

$\boxed{m = a(x_2^2 + x_1x_2 + x_1^2) + b(x_2 + x_1) + c}$

② $\dfrac{f'(x_1) + f'(x_2)}{2} = ?$ we know $f'(x) = 3ax^2 + 2bx + c$

$\Rightarrow \dfrac{f'(x_1) + f'(x_2)}{2} = \dfrac{3ax_1^2 + 2bx_1 + c + 3ax_2^2 + 2bx_2 + c}{2}$

$= \dfrac{3a(x_1^2 + x_2^2) + 2b(x_1 + x_2) + 2c}{2}$

$= \boxed{\dfrac{3a}{2}(x_1^2 + x_2^2) + b(x_1 + x_2) + c}$

Notice that for a cubic polynomial,

$$m \neq \frac{f'(x_1)+f'(x_2)}{2}$$

(like it did for a quadratic polynomial)

$\Rightarrow$ In general, we can only conclude that they're <u>not</u> the same. So, when asked to find the average rate of change, you need to find the slope through the 2 pts (<u>not</u> the avg of the derivatives). :)