5.1

Section 5.1 Eigenvectors and Eigenvalues

ELOs:

- Determine whether a vector is an eigenvector of a matrix; similarly, determine if a scalar is an eigenvalue of a matrix.
- Identify the eigenspace of a matrix A and relate it to the null space of the matrix $A - \lambda I$.
- Describe how eigenvalues are related to invertibility.

Motivating Example: The steady-state vectors for stochastic matrices described in Section 4.9 are a special case of the concept of eigenvectors and eigenvalues for $n \times n$ matrices in general.

Idea:

Eigenvectors of a matrix A are vectors on which the transformation described by A acts as a scalar.

- eigenvectors are vectors that just get "stretched" by A.

For example, consider the matrix transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

can think of λ as "stretching factor"

Definition 1 *An eigenvector of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $\boxed{A\mathbf{x} = \lambda \mathbf{x}}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there exists a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda \mathbf{x}$; such an $\mathbf{x}$ is called an eigenvector corresponding to λ.*

$\vec{x}$ = eigenvector

λ = eigenvalue

(λ is called "lambda")

Example: Show that 4 is an eigenvalue of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

Start here: $\begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 4\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and solve for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Example:

(a) Let

$$A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Is $\mathbf{u}$ an eigenvector of A? Is $\mathbf{v}$ an eigenvector of A? If so, find the associated eigenvalue.

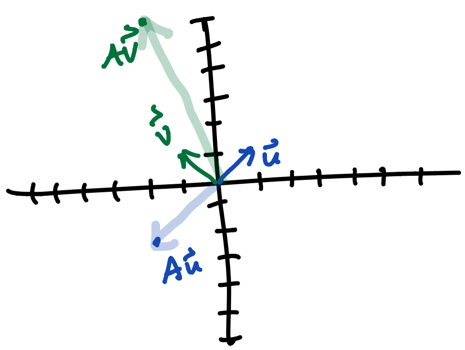
$$A\vec{u} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2\vec{u}$$

$$A\vec{v} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

(notice this is NOT a scaled version of $\vec{v}$)

Answer?

(b) Draw $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$. Then, draw $A\mathbf{u}$ and $A\mathbf{v}$ in $\mathbb{R}^2$. Describe the action of a linear transformation on its eigenvector(s).



Notice that geometrically $A\vec{u}$ gives back vector in direction of $\vec{u}$ (scaled) but $A\vec{v}$ gives back a totally differently oriented vector

Observation: λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation ______________ $\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

Hint: remember that λ is eigenvalue if

$$A\vec{x} = \lambda\vec{x} \iff A\vec{x} - \lambda\vec{x} = \vec{0} \iff (A - \lambda I)\vec{x} = \vec{0}$$

Definition 2 *The set of all solutions to the equation* $(A - \lambda I)\mathbf{x} = \mathbf{0}$ *is a subspace of* $\mathbb{R}^n$ *called the* *eigenspace* *of* A *corresponding to* λ. *Note: The eigenspace of* A *is the null space of the matrix* $A - \lambda I$.

(go back to last example & find eigenspace of A for $\lambda = -2$)

Example: Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$. An eigenvalue of A is $\lambda = 2$. Find a basis for the corresponding eigenspace.

start here:

① $A - \lambda I = A - 2I = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

② Solve $(A - \lambda I)\vec{x} = \vec{0}$ for $\vec{x}$.

③ Find basis for nullspace of $A - \lambda I$ (i.e. eigenspace of A corresponding to $\lambda = 2$).

Extra - Can we draw this eigenspace?

Exercise: Suppose λ is an eigenvalue of A. Determine an eigenvalue of $A^2, A^3, \ldots, A^n$. (assume A is $n \times n$)

We assume (because it's given to us) that $\exists \vec{x} \in \mathbb{R}^n$ s.t.

$A\vec{x} = \lambda\vec{x}$

Warning: An echelon form of a matrix A does not, in general, display the eigenvalues of A.

5.1

Theorem 1 *The eigenvalues of a triangular matrix are the entries on the main diagonal.*

This is really powerful & cool. It gives us one idea of how to find eigenvalues.

let's see how it works, by an example, rather than doing a proof.

Example: Let $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$. Find the eigenvalues of A.

$A - \lambda I = \begin{bmatrix} 3-\lambda & 6 & -8 \\ 0 & -\lambda & 6 \\ 0 & 0 & 2-\lambda \end{bmatrix}$

$(A-\lambda I)\vec{x} = \vec{0}$ has nontrivial solns iff there is at least one free variable i.e. if either (1) $3-\lambda=0$ or (2) $-\lambda=0$ or (3) $2-\lambda=0$

$\Rightarrow$ $\boxed{\lambda = 0, 2, 3 \text{ are e-values}}$

(a) If a matrix A has $\lambda = 0$ as an eigenvalue, what must be true about the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$?

(b) Based on your answer to part(a), is a matrix with a 0 eigenvalue invertible? Why or why not?

Theorem 2 *If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.*

Why? Qns to explore:

(1) could the zero vector be in $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$?

we know $A\vec{0} = \lambda\vec{0}$ is always true for any value of λ. So since we know all λ_i (for $i = 1, \ldots, r$) are distinct, then $\vec{0}$ cannot be an e-vector for only one e-value.

② could one of the e-vectors be a linear combo of the others?

Let's assume, as an example, that $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$

And remember that $\vec{v}_3$ corresponds to λ_3, $\vec{v}_2$ to λ_2, etc.

We know ① $A\vec{v}_3 = \lambda_3 \vec{v}_3$

But ② $A\vec{v}_3 = A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = \lambda_1\vec{v}_1 + \lambda_2\vec{v}_2$

(by our assumption)

This implies ① = ② or $\lambda_3\vec{v}_3 = \lambda_1\vec{v}_1 + \lambda_2\vec{v}_2$

$\Leftrightarrow \lambda_3(\vec{v}_1 + \vec{v}_2) = \lambda_1\vec{v}_1 + \lambda_2\vec{v}_2$

But this is a contradiction since $\vec{v}_1 \perp \vec{v}_2$.

$\Rightarrow$ Our assumption cannot be correct, i.e. $\vec{v}_3$ is independent of $\vec{v}_1$ & $\vec{v}_2$.

$\boxed{\text{Section 5.2 The Characteristic Equation}}$

ELOs:

- Find the eigenvalues of a square matrix using the characteristic equation.
- Define a similarity transform and explain how the eigenvalues of similar matrices are related.

Review: Eigenvectors and Eigenvalues

$A\vec{x} = \lambda\vec{x}$ $\quad$ $\vec{x}$ is called eigenvector

λ is called eigenvalue

Question: How do we find the eigenvectors $\mathbf{x}$ of A?

Solve $(A-\lambda I)\vec{x} = \vec{0}$ for nontrivial solns $\vec{x}$.

Question: How do we find the eigenvalues λ of A?

You find λ-values also when you solve $(A-\lambda I)\vec{x} = \vec{0}$ but the algebra can get messy... there's a better way!

Definition 1 *A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the* ***characteristic equation***

$$det(A - \lambda I) = 0$$

$p(\lambda) = det(A - \lambda I)$ is the n degree characteristic polynomial of $A_{n\times n}$.

Why is this true?

To find λ, solve $(A-\lambda I)\vec{x} = \vec{0}$ for nontrivial solns

$\Leftrightarrow$ $A-\lambda I$ is not invertible

$\Leftrightarrow$ $det(A-\lambda I)=0$.

Example: Find the eigenvalues of $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$.

$$A-\lambda I = \begin{bmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{bmatrix}$$

$$det(A-\lambda I) = (3-\lambda)(2-\lambda)-2=0$$

$$-\lambda^2-5\lambda+4=0$$

$$\lambda^2+5\lambda-4=0$$

$$(\lambda-4)(\lambda-1)=0 \Leftrightarrow \boxed{\lambda=1,4}$$

Notation:

- λ_j can be an eigenvalue for some (nonzero) eigenvector $\mathbf{x}$ if and only if λ_j is a root of the characteristic polynomial. That is, $p(\lambda_j) = 0$.
- The subspace of eigenvectors for the eigenvalue λ_j is called the λ_j **eigenspace** and denoted E_{λ_j}. The basis of eigenvectors is called the **eigenbasis for** E_{λ_j}.

(this example used in 5.3)

Example: Given $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3-\lambda \end{bmatrix}$$

(a) Find the eigenvalues, λ_j, of A.

$$\det(A-\lambda I) = (4-\lambda)\begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 2 & 3-\lambda \end{vmatrix} + 1\begin{vmatrix} 2 & -\lambda \\ 2 & -2 \end{vmatrix}$$

$$= (4-\lambda)(\lambda^2-3\lambda+2) + 2(6-2\lambda-2) + (-4+2\lambda)$$

$$= -\lambda^3+7\lambda^2-16\lambda+12 = (\lambda-2)(-\lambda^2+5\lambda-6) = (\lambda-2)(\lambda-2)(3-\lambda)$$

$$\Rightarrow \underbrace{\boxed{(\lambda-2)^2(3-\lambda)}}_{\text{characteristic polynomial for } A} = 0 \text{ gives solutions } \boxed{\lambda = 2, 3}$$

note: there are 2 distinct eigenvalues for A ($\lambda = 2$ has multiplicity 2 since factor $(\lambda-2)$ has exponent of 2 in characteristic polynomial)

(b) Find the corresponding eigenspaces, E_{λ_j}.

① for $\lambda = 2$

$$A - 2I = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A-2I)\vec{x} = \vec{0} \iff \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - \frac{1}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} x_3$$

$$\Rightarrow E_2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

notice A maps vectors from $\mathbb{R}^3$ to $\mathbb{R}^3$ and dimension of E_2 is 2 and dimension of E_3 is 1 .... 1+2=3.

② for $\lambda = 3$, we get (by same process)

$$E_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Observation: By collecting the eigenbases from all of the eigenspaces for a matrix $A_{n\times n}$, and putting them together, we may obtain a basis for $\mathbb{R}^n$. Under such conditions as those described in Theorem 2, we are able to understand the geometry of the transformation $T(\mathbf{x}) = A\mathbf{x}$ almost as well as if A were a diagonal matrix, and thus, we say that A is diagonalizable (to be continued in Section 5.3).

Reminder:

Theorem 2 *If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n\times n$ matrix A, then the set $\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$ is linearly independent.*

Example: Given $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) Find the characteristic equation of A.

(b) Find the eigenvalues of A.

Definition 2 *The degree n polynomial $\det(A - \lambda I)$ is called the characteristic polynomial. The (algebraic)* ***multiplicity*** *of an eigenvalue is its multiplicity as a root of the characteristic equation.*

Example: Given $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 9 & 1 & 3 & 0 \\ 1 & 2 & 5 & -1 \end{bmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A and the algebraic multiplicity of each eigenvalue.

Similarity

We call the map $A \longmapsto P^{-1}AP$ a $\boxed{\text{similarity}}$ transformation. 5.2

Definition 3 *For $n \times n$ matrices A and B, we say A is similar to B if there is an invertible matrix P such that*

$$P^{-1}AP = B \iff A = PBP^{-1}$$

Example: Show that if A and B are similar, then det(A)=det(B).

Note: If A is similar to B, then B is similar to A.

Theorem 3 *If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence, the same* eigenvalues *(with the same multiplicities).*

Proof: By defn, $B = P^{-1}AP$.

$\Rightarrow B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - \lambda P^{-1}P$ (since $P^{-1}P = I$)

$= P^{-1}(A)P - P^{-1}(\lambda I)P$

$= P^{-1}(A - \lambda I)P$

$\Rightarrow \det(B - \lambda I) = \det(P^{-1}(A-\lambda I)P) = \det(P^{-1})\det(A-\lambda I)\det(P)$

$= \det(P^{-1}P)\det(A-\lambda I) = \det(I)\det(A-\lambda I)$

$= \det(A - \lambda I)$ $\Leftrightarrow$ A & B have same characteristic polynomial (& same eigenvalues). #

Warning: Similar $\Rightarrow$ same eigenvalues, but same eigenvalues $\not\Rightarrow$ similar.

ex $\begin{bmatrix} 2 & 10 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ have same e-values but are NOT similar.

The Invertible Matrix Theorem (continued)

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ ____________________ matrix.

(c) A has ____________________ postions.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the ____________________ solution.

(e) The columns of A form a linearly ____________________ set.

(f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is ____________________.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has ____________________ solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A ____________________ $\mathbb{R}^n$.

(i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ ____________________ $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix C such that $CA =$ ____________________.

(k) There is an $n \times n$ matrix D such that $AD =$ ____________________.

(l) A^T is an ____________________ matrix.

(m) The ____________________ of A form a basis of __________.

(n) Col $A =$ __________

(o) $\dim(\text{Col } A) =$ __________

(p) rank $A =$ __________

(q) Nul $A =$ __________

(r) $\dim(\text{Nul } A) =$ __________

(s) __________ is not an eigenvalue of A.

(t) $det(A) \neq$ __________

<u>One last warning:</u>

Similarity is NOT the same as row equivalence!

☆ Row ops usually change eigenvalues. ☆

5.3

Section 5.3 Diagonalization

ELOs:

- Relate the similarity transform of a matrix to the eigenvectors of that matrix.
- Use eigenvalues and eigenvectors to diagonalize an $n \times n$ matrix A.
- Determine whether a matrix A is diagonalizable or not.

Goal: Develop a useful factorization $A = PDP^{-1}$ where P is the matrix of linearly independent eigenvectors of A and D is the diagonal matrix of corresponding eigenvalues.

So so cool that we can do this.

Warm-Up: Given $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors (eigenspace bases) of D.

(b) Find D^{10}.

(c) Find D^k where $k \in \mathbb{Z}^+$.

Example: Consider $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$.

(Note: this is one example we used in section 5.2.)

Let $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ be the matrix whose columns are the eigenvectors of A.

Let $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ be the diagonal matrix whose diagonal entries are the eigenvalues of A.

Check: $AP = PD$ $\Rightarrow A = PDP^{-1}$

Example: Using this factorization of A, find A^{100}.

$A^2 = AA = PD\underbrace{P^{-1}P}DP^{-1} = PDIDP^{-1} = PD^2P^{-1}$

$A^3 = AA^2 = PDP^{-1}PD^2P^{-1} = PDD^2P^{-1} = PD^3P^{-1}$

$\vdots$

$A^{100} = PD^{100}P^{-1}$

and since D is a diagonal matrix, D^{100} is quickly computed.

Definition 1 *A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix. That is,*

$$A = PDP^{-1}$$

where P is invertible and D is a diagonal matrix.

Note: *If there is a basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $\mathbb{R}^n$ (or $\mathbb{C}^n$) consisting of eigenvectors of A, then A is diagonalizable.*

Warning: *NOT all $n \times n$ matrices are diagonalizable!*

Theorem 5 *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.*

In fact, $\boxed{A = PDP^{-1}}$ *with diagonal matrix D if and only if the columns of P are n linearly independent eigenvectors of A. The diagonal entries of D are the eigenvalues of A that correspond, respectively, to the eigenvectors in P.*

A is diagonalizable if and only if there exists an eigenvector basis of $\mathbb{R}^n$.

Example: Diagonalize $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, if possible.

Step 1: Find the eigenvalues of A.

(Note: Generally, this step can be tedious. In general, you can use a computer to help with this.)

Step 2: Find the linearly independent eigenvectors of A.

Step 3: Construct P from the linearly independent eigenvectors found in Step 2. Note: this step might be impossible if there are fewer than 3 lin. indep. e-vectors (then you know A is <u>not</u> diagonalizable).

Step 4: Construct D from the corresponding eigenvalues found in Step 1.

Step 5: Check by verifying $AP = PD$.

Exercise: Show that $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is not diagonalizable.

(Hint: how many eigenvalues are there?)

Exercise: Is $C = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ diagonalizable? Justify your answer.

Theorem 6 *An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.*

Warning: This is a sufficient condition for a matrix to be diagonalizable, but it's not necessary.

n distinct evalues $\Rightarrow$ A is diagonalizable

BUT it is possible for a matrix without n distinct evalues to be diagonalizable.

(like example on pg 12)

Exercise: Diagonalize $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 24 & -12 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, if possible.

$A - \lambda I = \begin{bmatrix} -2-\lambda & 0 & 0 & 0 \\ 0 & -2-\lambda & 0 & 0 \\ 24 & -12 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix}$

$\det(A-\lambda I) = (-2-\lambda)^2(2-\lambda)^2 = 0$

$\iff \lambda = 2, -2$

$\lambda = 2$: $A - 2I = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 24 & -12 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x_1 = 0$, $x_2 = 0$, x_3, x_4 free

$(A-2I)\vec{x} = \vec{0} \iff \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4$

$\Rightarrow \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are 2 basis vectors for eigenspace E_2

$\lambda = -2$: $A + 2I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 24 & -12 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1/2 & 1/6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x_1 = \frac{1}{2}x_2 - \frac{1}{6}x_3$, $x_4 = 0$, x_2, x_3 free

$(A+2I)\vec{x} = \vec{0} \iff \vec{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1/6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\Rightarrow \vec{v}_3 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{v}_4 = \begin{bmatrix} -1/6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ are basis vectors for eigenspace E_{-2}

$\Rightarrow AP = PD$ s.t.

$P = \begin{bmatrix} 0 & 0 & 1/2 & -1/6 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

Theorem 7 *Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.*

(a) *For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.*

(b) *The matrix A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals n, which happens if and only if the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.*

(c) *If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.*

Section 5.4 Eigenvectors and Linear Transformations

ELOs:

- Recognize non-square matrices as transformations between dimensions.
- Construct a basis for a linear transformation T using eigenvectors.

Goal: Understand the matrix factorization $A = PDP^{-1}$ in terms of linear transformations.

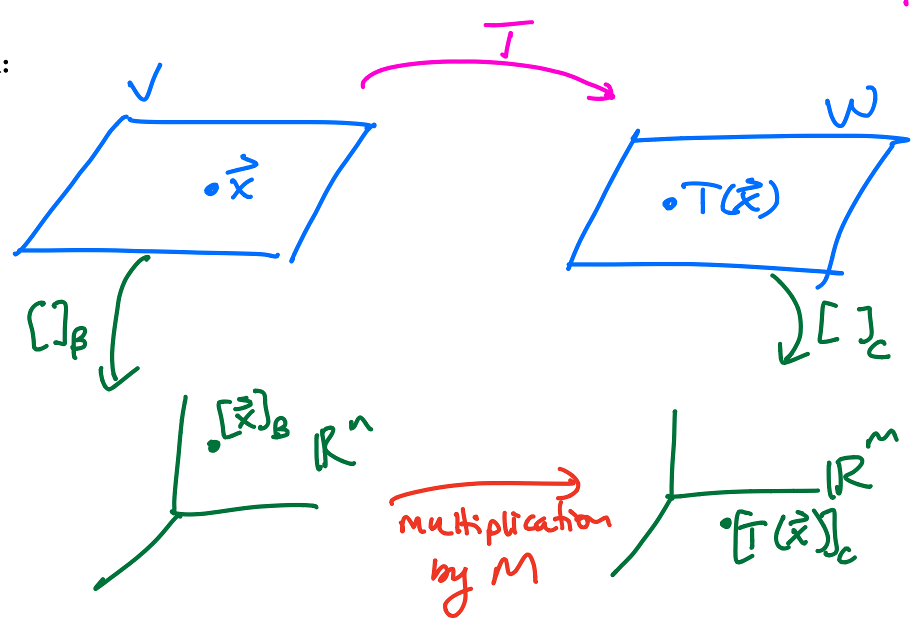
Warm-up: Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ be basis for $\mathbb{R}^2$.

(a) Given $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Find $\mathbf{x}$.

(b) Given $\mathbf{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{B}}$.

$T: V \to W$

Key Idea:



V, W finite-dim vector spaces

B = basis for V

C = basis for W

$M = \left[[T(\vec{b}_1)]_C \; [T(\vec{b}_2)]_C \; \cdots \; [T(\vec{b}_n)]_C \right]$ where $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$

M is the matrix for T relative to the bases B & C

Example: Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for V and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ is a basis for W. Let $T : V \to W$ be a linear transformation defined by its action on $\mathcal{B}$:

$$T(\mathbf{b}_1) = 3\mathbf{c}_1 - 2\mathbf{c}_2 + 5\mathbf{c}_3$$
$$T(\mathbf{b}_2) = 4\mathbf{c}_1 + 7\mathbf{c}_2 - \mathbf{c}_3.$$

$\dim(V) = 2$
$\dim(W) = 3$

Find the matrix M for T relative to $\mathcal{B}$ and $\mathcal{C}$.

$[T(\vec{b}_1)]_c = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$ $\quad [T(\vec{b}_2)]_c = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$ $\Rightarrow$ $\boxed{M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}}$

note: V and W are not same dimension so M is not square.

Example: Let $V = \mathbb{P}_3 = \text{span}\{1, t, t^2, t^3\}$, $W = \mathbb{P}_2 = \text{span}\{1, t, t^2\}$ and $T : V \longrightarrow W$ be the differential operator. Find the matrix of T with respect to the bases $\mathcal{B} = \{1, t, t^2, t^3\}$ in V and $\mathcal{C} = \{1, t, t^2\}$ in W.

One more example:

$T: \mathbb{P}_2 \to \mathbb{P}_2$ (notice it maps $\mathbb{P}_2$ to itself)

s.t. $T(a_0 + a_1 t + a_2 t^2) = 5a_0 + (a_0 - 2a_1)t + (8a_1 + 3a_2)t^2$.

Find matrix representation of T relative to standard basis $\mathcal{B} = \{1, t, t^2\}$.

$T(1) =$ $\Rightarrow [T(1)]_{\mathcal{B}} =$

$T(t) =$ $\Rightarrow [T(t)]_{\mathcal{B}} =$

$T(t^2)$ $\Rightarrow [T(t^2)]_{\mathcal{B}} =$

$\Rightarrow M = \begin{bmatrix} & & \\ & & \end{bmatrix}$ This is called $\mathcal{B}$-matrix for T.

Definition 1 *Two matrices A and B are* <u>*similar*</u> *if there is an invertible matrix P such that $B = P^{-1}AP$.*

Note: *Similar matrices arise when describing the same linear transformation with respect to different bases.* so cool!

<u>Example</u>: What if A is diagonalizable?

If A is diagonalizable, then $\exists$ Q and Q^{-1} s.t. $A = QDQ^{-1}$ and D is a diagonal matrix, where the columns of Q are the linearly ⫫ eigenvectors of A.

So if $A \sim B$ (similar to), then $B = P^{-1}AP$

$\Leftrightarrow B = P^{-1}(QDQ^{-1})P$

$\Leftrightarrow B = (P^{-1}Q)D(Q^{-1}P)$

$\Leftrightarrow B = (P^{-1}Q)D(P^{-1}Q)^{-1}$

i.e. B is also diagonalizable!

i.e. we can represent linear xform defined by A as multiplication by a diagonal matrix instead.

Theorem 8 Diagonal Matrix Representation
Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$-matrix for the transformation of $\mathbf{x} \mapsto A\mathbf{x}$.

Example: Consider $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$. In Section 5.2, we found $E_{\lambda=4} = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ and $E_{\lambda=1} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$. Find $M = [T]_{\mathcal{B}}$.

$\Rightarrow P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ $D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ $P^{-1} = \frac{1}{-3} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}$

$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ $\Rightarrow$ $\boxed{M = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}}$

Note: $\vec{x} \mapsto A\vec{x}$ and $\vec{x} \mapsto D\vec{x}$ describe same linear xform relative to different bases

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have the standard transformation matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ such that the $\mathcal{B}$-matrix for T is diagonal.

$\det(A - \lambda I) = (7-\lambda)(1-\lambda) + 8$
$= 15 - 8\lambda + \lambda^2$
$= (\lambda-3)(\lambda-5) = 0$
$\Rightarrow \lambda = 3, 5$

$\lambda = 3$: evector is $\vec{x}_1 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

$\lambda = 5$: evector is $\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\Rightarrow$ $\boxed{D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}}$

$\Rightarrow$ basis for $\mathbb{R}^2$ s.t. $\mathcal{B}$-matrix is D $= \boxed{\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}}$

Example: Let $A = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}$.

(a) Check that A is not diagonalizable using the eigenspace dimension theorem. (i.e. look at $\dim(\text{Nul}(A - \lambda I))$)

$\det(A - \lambda I) = 0$ yields only $\lambda = -2$

and for $\lambda = -2$ we get only the eigenvector $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$

$\Rightarrow A$ is not diagonalizable w/ eigenvalues since P^{-1} DNE

(b) Let $P = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$. Find the $\mathcal{B}$-matrix , $P^{-1}AP$, a almost triangular matrix.

This matrix is called the **Jordan form** of A.

$P^{-1} = \frac{1}{3-4} \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} = -1 \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$

$\Rightarrow B = P^{-1}AP = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$

close as possible to being diagonal.

Note: Every square matrix A is similar to a matrix in Jordan form.

We could say a lot more about Jordan form matrices ... but we're moving on.

5.5 Complex Eigenvalues

* We won't do this section in detail but I want to at least give you an overview of the ideas.

Overall idea: The eigenvalues (of a matrix w/ all $\mathbb{R}$-valued elements) can be complex #s.

ex $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ which is a rotation matrix, clockwise by 90°.

This does not send any vectors to multiples of themselves! $\Rightarrow$ no real eigenvalues.

But we can still find characteristic polynomial and look for eigenvalues.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i$$

eigenvectors DO exist IF A acts on $\mathbb{C}^2$ instead of $\mathbb{R}^2$

$\lambda = i$ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i\begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\lambda = -i$ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i\begin{bmatrix} 1 \\ i \end{bmatrix}$

$\Rightarrow$ eigenvectors are $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$