Real & Complex Analysis Qualifying Exam Solution, Fall 2007

Shiu-Tang Li

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A-1

Apply dominated convergence theorem with dominating function $|f|$ on every $h_n \to 0$ to prove $F(x+h_n) \to F(x)$. Arbitrariness of $\{h_n\}$ implies that F is continuous on x for every $x \in \mathbb{R}$.

A-2

We have $\|Mg\|_2 = (\int_{\mathbb{R}} f^2 g^2 \, dx)^{1/2} \leq \|f\|_\infty (\int_{\mathbb{R}} g^2 \, dx)^{1/2} = \|f\|_\infty \|g\|_{L^2(\mathbb{R})}$.

To see $\|M\| \geq \|f\|_\infty$, consider $g_\epsilon = sgn(f)\chi_{[-n,n]}\chi_{\{|f|>\|f\|_\infty - \epsilon\}}$, where $n = n_\epsilon$ is chosen s.t. $\mu(|f| > \|f\|_\infty - \epsilon, x \in [-n,n]) > 0$.

Therefore, $\|Mg_\epsilon\|_2^2 = \int_{-n}^n |f|^2 \chi_{\{|f|>\|f\|_\infty - \epsilon\}} \, d\mu \geq (\|f\|_\infty - \epsilon)^2 \mu(|f| > \|f\|_\infty - \epsilon, x \in [-n,n]) = \|g_\epsilon\|_{L^2}^2 (\|f\|_\infty - \epsilon)^2$. Since ϵ is arbitrary, we have $\|M\| \geq \|f\|_\infty$.

A-3

(a) Let $Tv = \lambda v, v \neq 0$. Since $(Tv, v) = \lambda(v,v) = (v, Tv) = \overline{\lambda}(v,v)$ and $(v,v) > 0$, λ is real.

(b) Let $Tu = \mu u$, $Tv = \lambda v$, $\mu \neq \lambda$ and $u, v \neq 0$. We have $(Tu, v) = \mu(u,v) = (u, Tv) = \lambda(u,v)$. $\mu \neq \lambda \Rightarrow (u,v) = 0$.

(c) For each λ_n, we pick some $\|x_n\| = 1$ so that $Tx_n = \lambda_n x_n$. We claim that $X = \{\sum_{n=1}^\infty a_n x_n : \sum_{n=1}^\infty |a_n|^2 < \infty\}$ is a closed subspace of H. For if

$y_k := \sum_{n=1}^{\infty} a_{kn}x_n$ is Cauchy, $a_{kn} \to b_n$ as $k \to \infty$ for each $n \in \mathbb{N}$. Given $\epsilon > 0$, we have $\|y_m - y_n\| < \epsilon$ for all $m, n > K$, and now we apply Fatou's lemma to get

$$\sum_{n=1}^{\infty} |b_n - a_{Kn}|^2 \leq \liminf_{k\to\infty} \sum_{n=1}^{\infty} |a_{kn} - a_{Kn}|^2$$
$$= \liminf_{k\to\infty} \|y_k - y_K\| \leq \epsilon.$$

Besides, we may pick K large so that $\Big(\sum_{n=1}^{\infty} |b_n|^2\Big)^{1/2} \leq \Big(\sum_{n=1}^{\infty} |b_n - a_{Kn}|^2\Big)^{1/2} + \Big(\sum_{n=1}^{\infty} |a_{Kn}|^2\Big)^{1/2} < \infty$. That is, $\sum_{n=1}^{\infty} b_n x_n \in X$. That $y_k \to \sum_{n=1}^{\infty} b_n x_n$ in X is equivalent to the fact that X is closed.

Since X is closed, we may decompose $H = X \oplus X^{\perp}$. We claim that $x_n \rightharpoonup 0$. First, for each $y \in H$ we may write it as $\sum_{m=1}^{\infty} a_m x_m + x'$, where $\sum_{m=1}^{\infty} a_m x_m \in X$ and $x' \in X^{\perp}$. We have

$$(x_n, y) = (x_n, \sum_{m=1}^{\infty} a_m x_m + x') = (x_n, \sum_{m=1}^{\infty} a_m x_m)$$
$$= (x_n, \sum_{m=1}^{n} a_m x_m) + (x_n, \sum_{m=n+1}^{\infty} a_m x_m)$$
$$= a_n + (x_n, \sum_{m=n+1}^{\infty} a_m x_m) \to 0 \text{ as } n \to \infty.$$

The above convergence is due to $\sum_{m=1}^{\infty} |a_m|^2 < \infty$. We have shown our claim.

Next, since $\{x_n\}_n$ is a bounded sequence, $\{Tx_n\}_n$ is precompact in H. We claim that $Tx_n \to 0$. If not, by precompactness of $\{Tx_n\}_n$ we can find a subsequence $\{Tx_{m_k}\}_k$ so that $Tx_{m_k} \to z \neq 0 \Rightarrow Tx_{m_k} \rightharpoonup z \neq 0$. However, $x_n \rightharpoonup 0$ implies $Tx_n \rightharpoonup 0$ by that T is adjoint, which is a contradiction. Therefore,

$$Tx_n \to 0 \Rightarrow \|Tx_n\| \to 0$$
$$\Rightarrow \|\lambda_n x_n\| \to 0$$
$$\Rightarrow \|\lambda_n\| \to 0$$
$$\Rightarrow \lambda_n \to 0.$$

And the proof is complete.

A-4

See the sol $A-4$ in spring 2008 real and complex analysis qualifying exam. (Or directly from Rudin's book.)

A-5

$$\begin{aligned}\|Tg\|_2 = (\int_0^1 (Tg(x))^2\,dx)^{1/2} &\le (\int_0^1 (\int_0^1 K(x,y)^2\,dy)(\int_0^1 f(y)^2\,dy)\,dx)^{1/2} \\ &= \|f\|_{L^2(I)} \int_I \int_I K(x,y)^2\,dy\,dx \\ &= \|f\|_{L^2(I)} \int_{I\times I} K(x,y)^2\,d(x\times y)\end{aligned}$$

where the last identity is due to Tonelli's theorem.

B-6

I would prove further that if $f(z) \le A + B|z|^n$ for some $A \ge 0, B > 0$ and for all $z \in \mathbb{C}$, then $f(z)$ is a polynomial of degree $\le n$.

we first note that $g(z) := \frac{f(z)-f(0)}{z}$ for $z \neq 0$ and $g(0) = f'(0)$ is also an entire function , for g is holomorphic on $\mathbb{C}\setminus\{0\}$ and is continuous on $\mathbb{C}$, thus we may prove holomorphicity of g on $\mathbb{C}$ using Morera's theorem.

The proof is by induction. When $n = 0$, this is Liouville's theorem. For $n = N > 1$, assume that $f(z) \le A + B|z|^N$, we have $|g(z)| \le \frac{A+B|z|^N+|f(0)|}{|z|} \le A' + B|z|^{N-1}$ when $|z| > 1$, and $|g(z)| \le A'' + B|z|^{N-1}$ when $|z| \le 1$ since $|g|$ is bounded on bounded domains. Therefore, by induction hypothesis, $g(z)$ is a polynomial of degree $\le N-1$. Since $g(z)z + f(0) = f(z)$ for every $z \in \mathbb{C}$, f is a polynomial of degree $\le N$, and this completes the proof.

B-7

(a) Let $\phi(z) = \frac{i-z}{i+z}$, which is analytic on H, and for $z = a+bi$, $b > 0$, $|i-z| = |-a+(1-b)i| < |a+(1+b)i| = |i+z|$, which shows that $\phi : H \to D := \{z : |z| < 1\}$. To see $\phi(z)$ is onto and $1-1$, for any

$w = re^{i\theta} \in D$, $r < 1$, $w = \frac{i-z}{i+z} \Rightarrow z = i\frac{1-w}{1+w} = i\frac{1-re^{i\theta}}{1+re^{i\theta}} = i\frac{1-r^2-2i\sin(\theta)}{1+r^2+2r\cos(\theta)} \in H$. Thus, we may define the inverse map of ϕ by $\phi^{-1}(z) = i\frac{1-z}{1+z} : D \to H$. These facts show that ϕ is a conformal map from H to D, which takes i to 0.

(b) Consider the holomorphic map $\phi \circ f \circ \phi^{-1} : D \to D$.

$$\begin{aligned}
|(\phi \circ f \circ \phi^{-1})'(0)| &= |\phi'(f(\phi^{-1}(0)))f'(\phi^{-1}(0))(\phi^{-1})'(0)| \\
&= |\phi'(i)f'(i)(\phi^{-1})'(0)| \\
&= |\phi'(i)f'(i)\frac{1}{\phi'(\phi^{-1}(0))}| \\
&= |\phi'(i)f'(i)\frac{1}{\phi'(i)}| = |f'(i)|.
\end{aligned}$$

In the above lines we use the fact that $|\phi'(i)| \neq 0$, for ϕ is an injective holomorphic map. By Schwarz's lemma, we have $1 \geq |(\phi \circ f \circ \phi^{-1})'(0)| = |f'(i)|$. If the equality holds, then $\phi \circ f \circ \phi^{-1}(z) = e^{i\theta}z$, and thus

$$\begin{aligned}
f(z) &= \phi^{-1} \circ e^{i\theta}z \circ \phi(z) \\
&= \phi^{-1} \circ e^{i\theta}\frac{i-z}{i+z} \\
&= i\frac{1 - e^{i\theta}\frac{i-z}{i+z}}{1 + e^{i\theta}\frac{i-z}{i+z}} \\
&= i\frac{i + z - e^{i\theta}(i-z)}{i + z + e^{i\theta}(i-z)} \\
&= i\frac{e^{-i\theta/2}(i+z) - e^{i\theta/2}(i-z)}{e^{-i\theta/2}(i+z) + e^{i\theta/2}(i-z)} \\
&= \frac{1}{-i} \cdot \frac{-2i\sin(\theta/2)\cdot i + 2\cos(\theta/2)z}{2\cos(\theta/2)\cdot i - 2i\sin(\theta/2)z} \\
&= \frac{\sin(\theta/2) + \cos(\theta/2)z}{\cos(\theta/2) - \sin(\theta/2)z}.
\end{aligned}$$

B-8

Define $g(z) = f(z)$ when $0 \leq Re\, z \leq 1$. Define $g(z) = \overline{g(2-\overline{z})}$ when $1 < Re\, z \leq 2$. We claim that $g(z)$ is holomorphic on $R_{1,2} := \{z : 1 < Re\, z < 2\}$. To see this, for any $z_0 \in R_{1,2}$, there is some $\epsilon > 0$ so that $B_{z_0}(\epsilon) \subset R_{1,2}$. Since $g(z)$ is holomorphic on $R_{0,1} := \{z : 0 < Re\, z < 1\}$, we have $\overline{g(2-\overline{z})} = \overline{\sum_{n=0}^{\infty} a_n(2-\overline{z}-(2-\overline{z_0}))^n} = \sum_{n=0}^{\infty} \overline{a_n}(-1)^n(z-z_0)^n$, which proves the existence of power series at every neighborhood of $z \in R_{1,2}$ and hence our claim.

Since $g(z)$ defined in this way is continuous on $\{z : 0 < Re\, z < 2\}$ due to the fact $f(1+ix) \in \mathbb{R}$ for ever $x \in \mathbb{R}$, and $g(z)$ holomorphic on $R_{0,1} \cup R_{1,2}$, by Morera's theorem, $g(z)$ is holomorphic on $\{z : 0 < Re\, z < 2\}$. Also, from our definition of g, $g(ix) = g(2+ix)$ for every $x \in \mathbb{R}$, for $f(ix) \in \mathbb{R}\ \forall x \in \mathbb{R}$.

Now for every $2n \leq Re\, z \leq 2n+2$, $n \in \mathbb{Z}$, we may define $g(z) = g(z-2n)$. It is straightforward that g is holomorphic on $2n < Re\, z < 2n+2$. In addition, it is continuous on $\{z : Re\, z = 2n, n \in \mathbb{Z}\}$. By Morera's theorem, $g(z)$ is holomorphic on $\mathbb{C}$.

Since f and g coincide on $R_{0,1}$, $f \equiv g$ by the uniqueness theorem. As a result, for any $z \in \mathbb{C}$, $f(z) = g(z) = g(z+2) = f(z+2)$.

B-9

For each $z \in \Omega$, there is an ϵ-ball centered at z and its closure is contained in Ω. Let C be its boundary. By Cauchy's integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta \\ \Rightarrow f'(z) &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta, \end{aligned}$$

for which the proof is omitted. Therefore, for ever $f \in \mathcal{F}$, $|f'(z)| \leq \frac{1}{2\pi} \cdot \frac{M}{\epsilon^2} \cdot 2\pi\epsilon := B_z$, where M is chosen that $|f(w)| < M$ for all $w \in \overline{B_z(\epsilon)}$.

B-10

Let $C_R = Re^{i\theta}$, θ goes from 0 to π. By the residue theorem, $\int_{-R}^{R} \frac{\cos(x)}{x^2+4}\,dx + i\int_{-R}^{R} \frac{\sin(x)}{x^2+4}\,dx + \int_{C_R} \frac{e^{iz}}{z^2+4}\,dz = 2\pi i Res(f; 2i) = 2\pi i \cdot \frac{e^{-2}}{4i} = \frac{\pi}{2}e^{-2}$. Since $|\int_{C_R} \frac{e^{iz}}{z^2+4}\,dz| \leq \pi R \cdot \frac{1}{R^2+4}$, let $R \to \infty$ we have $\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+4}\,dx = \frac{\pi}{2}e^{-2}$.