Probability Qualifying Exam Solution, Fall 2009

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(1) For each $B \in \mathscr{B}(\mathbb{R})$, write it as $B = \{B \cap (t, \infty)\} \cup \{B \cap (-\infty, t]\}$.

Let $Y = X \wedge t$; note that $\{X \geq t\} = \{Y = t\}$;

$$\begin{aligned}
\int_{Y^{-1}(B)} X dP &= \int_{Y^{-1}(B\cap(-\infty,t])} X dP + \int_{Y^{-1}(B\cap(t,\infty))} X dP \\
&= \int_{Y^{-1}(B\cap(-\infty,t])} X dP \\
&= \int_{Y^{-1}(B\cap(-\infty,t))} X dP + \int_{Y^{-1}(B\cap\{t\})} X dP \\
&= \int_{Y^{-1}(B\cap(-\infty,t))} Y dP + 1_B(t) \int_{\{X\geq t\}} X dP \\
&= \int_{Y^{-1}(B\cap(-\infty,t))} Y dP + 1_B(t) \int_t^\infty u e^{-u} du \\
&= \int_{Y^{-1}(B\cap(-\infty,t))} Y dP + 1_B(t) e^{-t}(t+1) \\
&= \int_{Y^{-1}(B\cap(-\infty,t))} Y dP + 1_B(t) \int_{\{X\geq t\}} (t+1) dP \\
&= \int_{Y^{-1}(B)} Y 1_{\{Y<t\}} dP + \int_{Y^{-1}(B\cap\{t\})} (t+1) dP \\
&= \int_{Y^{-1}(B)} Y 1_{\{Y<t\}} dP + \int_{Y^{-1}(B)} (t+1) 1_{\{Y=t\}} dP
\end{aligned}$$

(2) Similar to (1).

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Let $I = [a, b]$. $Cov(f(X), g(X)) = E[f(X)g(X)] - E[f(X)]E[g(X)] = \int_a^b f(x)g(x)\,dF(x) - \int_a^b f(x)\,dF(x) \int_a^b g(x)\,dF(x)$.

Assume first f and g are simple functions. We may write $f(x) = \sum_{i=1}^n a_i 1_{(t_{i-1},t_i]}$, $g(x) = \sum_{i=1}^n b_i 1_{(t_{i-1},t_i]}$, where $a_{i+1} \geq a_i$ and $b_{i+1} \geq b_i$ for all i, and $0 = t_0 < t_1 < \cdots < t_n = 1$.

$$
\begin{aligned}
&\int_a^b f(x)\,dF(x) \int_a^b g(x)\,dF(x) \\
&= \sum_{i=1}^n a_i(F(t_i) - F(t_{i-1})) \sum_{j=1}^n b_j(F(t_j) - F(t_{j-1})) \\
&= \sum_{j=1}^n \sum_{i=1}^n a_i(F(t_i) - F(t_{i-1}))b_j(F(t_j) - F(t_{j-1})) \\
&= \sum_{j=1}^n \sum_{i=1}^n a_i(F(t_i) - F(t_{i-1}))(b_j - b_i)(F(t_j) - F(t_{j-1})) \\
&+ \sum_{j=1}^n \sum_{i=1}^n a_i(F(t_i) - F(t_{i-1}))b_i(F(t_j) - F(t_{j-1})) \\
&= \sum_{0 \leq i \neq j \leq n} a_i(F(t_i) - F(t_{i-1}))(b_j - b_i)(F(t_j) - F(t_{j-1})) + \sum_{i=1}^n a_i b_i(F(t_i) - F(t_{i-1})) \\
&= \frac{1}{2} \sum_{0 \leq i \neq j \leq n} (a_i - a_j)(b_j - b_i)(F(t_i) - F(t_{i-1}))(F(t_j) - F(t_{j-1})) + \sum_{i=1}^n a_i b_i(F(t_i) - F(t_{i-1})) \\
&\leq \sum_{i=1}^n a_i b_i(F(t_i) - F(t_{i-1})) = \int_a^b f(x)g(x)\,dF(x)
\end{aligned}
$$

Passing limit from simple functions to increasing functions to obtain the result.

3

Step 1. We'd like to show $\sum_{n=1}^\infty \frac{X_n+1}{n}$ exists and is finite a.s.

To check this random series converges, it suffices to show the following 3 series all converge: (Let $Y_n = \frac{X_n+1}{n} \cdot 1_{\{|\frac{X_n+1}{n}| \leq 1\}}$)

(1)$\sum_{n=1}^{\infty} P(|\frac{X_n+1}{n}| > 1) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

(2)$\sum_{n=1}^{\infty} E[Y_n] = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

(3)$\sum_{n=1}^{\infty} \sigma^2(Y_n) = \sum_{n=1}^{\infty}(E[Y_n^2] - E[Y_n]^2) = \sum_{n=1}^{\infty}(\frac{1}{n^2} - \frac{1}{n^4}) < \infty$

As a result, $\sum_{n=1}^{\infty} \frac{X_n+1}{n}$ converges a.s. by Kolmogorov's 3 series theorem.

Step 2. By Kronecker's lemma, which states that if $\{x_k\}$ is a sequence of reals and $\{a_k\}$ is an nondecreasing sequence of positive numbers that converges to ∞, then $\sum_{k=1}^{\infty} \frac{x_k}{a_k}$ converges $\Rightarrow \frac{1}{a_n} \sum_{k=1}^{n} x_k \to 0$ when $n \to \infty$.

Thus, $\sum_{k=1}^{\infty} \frac{X_k+1}{k}$ converges a.s. $\Rightarrow \frac{1}{n} \sum_{k=1}^{n}(X_k+1) = 1 + \frac{S_n}{n} \to 0$ a.s. $\Rightarrow \frac{S_n}{n} \to -1$ a.s.

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$E[N_m] = E[E[N_m|N_{m-1}]] = E[\sum_{k=1}^{\infty} 1_{\{N_{m-1}=k\}} p \times (k+1) + (1-p) \times (k + 1 + E[N_m])] \Rightarrow p \times E[N_m] = 1 + E[N_{m-1}]$. Use math induction.

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Let $\{E_{n,i}\}$ be the event that the $i-th$ card is in position i in a deck of n cards.

$$
\begin{aligned}
&P(E_n) = P(\bigcup_{i=1}^{n} E_{n,i}) \\
&= \sum_{i=1}^{n} P(E_{n,i}) - \sum_{1 \le i < j \le n} P(E_{n,i} \cap E_{n,j}) + \sum_{1 \le i < j < k \le n} P(E_{n,i} \cap E_{n,j} \cap E_{n,k}) - \cdots \\
&= n \times \frac{1}{n} - \binom{n}{2} \frac{1}{n^2} + \binom{n}{3} \frac{1}{n^3} - \cdots \\
&\to 1 - e^{-1} \qquad \text{when } n \to \infty
\end{aligned}
$$

6

(1)Conv. in distribution:
$\phi_{S_n/n}(t) = (\phi_{X_1/n}(t))^n = \phi_{X_1}(t/n))^n = (e^{-|t/n|})^n = e^{-|t|}$ for all n.

(2)Not conv. in pr.:
Let $Z_n = \frac{S_n}{n} - \frac{S_{n-1}}{n-1} = \frac{(n-1)X_n - S_{n-1}}{n(n-1)}$. $\phi_{Z_n}(t) = e^{-|\frac{t}{n}|} \times (e^{-|\frac{t}{n(n-1)}|})^{n-1} = e^{-|\frac{t}{n/2}|}$, showing that $P(|Z_n| > 2012) > 1/2012$ for all n large, which shows $\{\frac{S_n}{n}\}_n$ is not Cauchy in pr.

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(1)Let $F_n = \sigma(X_1, \cdots, X_n)$. We have

$$\begin{aligned}
&E[X_1X_2\cdots X_n|Fn-1] \\
&= X_1X_2\cdots X_{n-1}E[X_n|F_{n-1}] \\
&= X_1X_2\cdots X_{n-1}E[X_n] \\
&= X_1X_2\cdots X_{n-1}.
\end{aligned}$$

(2)Since

$$\begin{aligned}
&E[|lnX_1|] \\
&= -\frac{1}{2}\int_0^1 lnx\,dx + \frac{1}{2}\int_1^2 lnx\,dx \\
&= -\frac{1}{2}(xlnx - x)|_0^1 + \frac{1}{2}(xlnx - x)|_1^2 \\
&= \frac{1}{2} + \frac{1}{2}(2ln2 - 1) = ln2 < \infty
\end{aligned}$$

and $E[lnX_1] = ln2 - 1 < 0$, and $lnX_1, lnX_2, \cdots$, are i.i.d, by strong law of large numbers, we have $\frac{\sum_{i=1}^n lnX_i}{n} \to E[lnX_1] = ln2 - 1$. Therefore, for a.s. ω we have $\frac{ln(\Pi_{i=1}^n X_i(\omega))}{n} \to ln2 - 1$. For this ω, we may choose $N(\omega)$ s.t. for all $n > N(\omega)$, we have $\frac{ln(\Pi_{i=1}^n X_i(\omega))}{n} - (ln2-1) \le \left|\frac{ln(\Pi_{i=1}^n X_i(\omega))}{n} - (ln2-1)\right| < \frac{1-ln2}{2}$. Therefore,

$$\frac{ln(\Pi_{i=1}^n X_i(\omega))}{n} < \frac{1-ln2}{2} + ln2 - 1 = \frac{ln2-1}{2}$$

for all $n > N(\omega)$. This implies $\Pi_{i=1}^n X_i(\omega) < e^{n(ln2-1)/2}$ for all $n > N(\omega)$, and hence $\lim_{n\to\infty} \Pi_{i=1}^n X_i(\omega) = 0$.

This is what martingale convergence theorem has told us: L^1 bounded submartingale converges a.s. to a finite limit.

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$$\begin{aligned}&\phi_{X_1}(t)\\&=\phi_{Y_1}(t/\sqrt{2})\phi_{Y_2}(t/\sqrt{2}) \text{ by independence of } Y_1 \text{ and } Y_2\\&=e^{-t^2/4}\times e^{-t^2/4}=e^{-t^2/2}\end{aligned}$$

on the other hand,

$$\begin{aligned}&\phi_{X_2}(t)\\&=\phi_{Y_1}(t/\sqrt{2})\phi_{Y_2}(-t/\sqrt{2}) \text{ by independence of } Y_1 \text{ and } Y_2\\&=e^{-t^2/4}\times e^{-t^2/4}=e^{-t^2/2}\end{aligned}$$

Therefore, both X_1 and X_2 are normally distributed with mean 0 and variance 1.

9

By Cauchy-Schwarz inequality, $E[X]^2 = E[X1_{\{X>0\}}]^2 \le E[X^2]E[1^2_{\{X>0\}}] = E[X^2]P(X > 0) = E[X^2] - E[X^2]P(X = 0)$, we have $P(X = 0) \le \frac{E[X^2]-E[X]^2}{E[X^2]}$, as desired.

10

Using change of variable formula to compute the density. Let $U = X/Y, V = Y \Rightarrow X = UV, Y = V$. we have the new joint density of U, V: $f_{U,V}(u, v) = f_{X,Y}(x(u, v), y(u, v))\times|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}| = \frac{1}{2\pi}e^{-u^2v^2/2}e^{-v^2/2}|\begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix}|$,
As a result, $f_U(u) = \int_{\mathbb{R}} f_{U,V}(u, v)\, dv = 2\int_0^\infty \frac{1}{2\pi}e^{-u^2v^2/2}e^{-v^2/2}v\, dv$
$= \frac{1}{\pi}\cdot\frac{-1}{1+u^2}e^{-(1+u^2)v^2/2}|_{v=0}^{v=\infty} = \frac{1}{\pi}\cdot\frac{1}{1+u^2}$.

Thus $P(X/Y \le x) = \int_{-\infty}^x \frac{1}{\pi}\cdot\frac{1}{1+u^2}\, du = \frac{1}{\pi}\tan^{-1}(u)|_{-\infty}^x = \frac{1}{\pi}(\tan^{-1}(x) + \frac{\pi}{2})$.

11

Since the random series converges absolutely a.s., it converges a.s. to a finite limit, say X. Thus the sequence also converges weakly to X. The characteristic function $\phi_N(t)$ for $\sum_{n=1}^{N} \frac{X_n}{2^n}$ is

$$\begin{aligned}\phi_N(t) &= \Pi_{n=1}^{N} \frac{1-e^{\frac{it}{2^n}}}{2} \\ &= \frac{1}{2^N}(1+e^{\frac{it}{2^N}}+e^{\frac{i2t}{2^N}}+\cdots+e^{it}) \\ &= \frac{1}{2^N}\cdot\frac{1-e^{it}}{1-e^{\frac{it}{2^N}}},\end{aligned}$$

where $\frac{1}{2^N}\cdot\frac{1-e^{it}}{1-e^{\frac{it}{2^N}}} \to \frac{1-e^{it}}{-it}$ as $N \to \infty$ using L'Hopital's rule. $\frac{1-e^{it}}{-it}$ is exactly the characteristic function for uniform distribution on (0,1).