Math 5440
Friday 9/3

We'll go over the multivariable FTC and its consequences in $\mathbb{R}^n$, including the $\mathbb{R}^n$ divergence theorem and the $\mathbb{R}^2$ Green's Theorem (which implies Stoke's Theorem for surfaces $S^2 \subset \mathbb{R}^3$). We'll use Wednesday notes.

It's worth noting that the $\mathbb{R}^n$ FTC is also the basic building block for the generalized Stokes' Theorem in pure math,

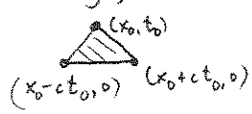
$$\underbrace{\int_{\partial M}}_{\text{bdry of an n-manifold}} \underbrace{\omega}_{\text{an n-1 form}} = \int_M \underbrace{d\omega}_{\text{the n-form which is the differential of the n-1 form M.}}$$

This generalized Stokes' Theorem is a key tool in many areas of pure mathematics (& theoretical physics)
(wikipedia Stokes' Theorem)

Homework for Friday Sept 10:

(1) We've worked hard to derive the solution

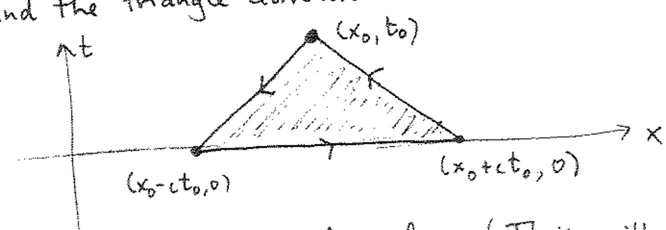
$$u(x_0,t_0) = \frac{1}{2}\left(f(x_0-ct_0)+f(x_0+ct_0)\right) + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(x)\,dx + \frac{1}{2c}\iint F\,dA$$



to the IVP

$$\begin{cases} u_{tt}-c^2u_{xx} = F(x,t) & x\in\mathbb{R},\ t>0 \\ u(x,0) = f(x) & x\in\mathbb{R} \\ u_t(x,0) = g(x) & \text{''} \end{cases}$$

Apply $\boxed{\text{Green's Theorem}}$ with the vector field $\langle M,N\rangle = \langle u_t, c^2u_x\rangle$
and the triangle domain in $\mathbb{R}^2$



to derive the same solution formula. (This will be fun, force you to refresh your line integral skills, and to notice that along the diagonal sides of the triangle the vector field is actually a multiple of the directional derivative of u in those directions.)

2 Let $\Omega \subset \mathbb{R}^n$ an open domain, $f: \Omega \to \mathbb{R}$ continuous.

Suppose that for all open balls $B(x_0, r) = \{x \in \mathbb{R}^n \text{ s.t. } \|\vec{x} - \vec{x}_0\| < r\}$ with $B(x_0, r) \subset \Omega$, we have

$$* \int_{B(x_0,r)} f(x)\, dV = 0 \qquad \begin{pmatrix} \text{in } \mathbb{R}, & dV = dx \\ \text{in } \mathbb{R}^2 & dV = dA = dx\,dy \\ \text{in } \mathbb{R}^3 & dV = dx\,dy\,dz \\ & \text{etc.} \end{pmatrix}$$

Prove $f(x) \equiv 0$ in Ω.

Hint: Assume $f(x_0) \neq 0$. Use f cont. @ x_0 to derive a contradiction to $*$

3 We worked hard in an earlier HW to derive the conservation of mass (continuity eq'n) and conservation of momentum equations for waves in friction-free compressible fluids, like idealized gases. We did this for plane waves in $\mathbb{R}^3$ (with motion only in the x-direction. Recall, the equations were (for density $\rho(x,t)$ and x-velocity $u(x,t)$), and body force/mass F:

$$(1.13) \quad \begin{cases} \rho u_x + u\rho_x + \rho_t = 0 \\ \rho u_t + \rho u u_x + p_x = \rho F \end{cases}$$

We linearized these to get

$$(1.14) \quad \begin{cases} \rho_0 u_x + \rho_t = 0 \\ \rho_0 u_t + p'(\rho_0)\rho_x = \rho_0 F \end{cases}$$

which led to 1-d wave equations for $\rho(x,t)$, $u(x,t)$.

There are analogous systems of PDE's, called the $\boxed{\text{Navier-Stokes'}}$ equations (for 1.13)), for arbitrary waves in $\mathbb{R}^3$, with analogous linearizations which lead to $\mathbb{R}^n$ versions of the wave equation, i.e. for ρ and for the components of the velocity vector. Your job in this problem is to derive NS, using the $\boxed{\text{divergence theorem}}$, $\boxed{\text{cons. of mass, Newton's 2nd law, \& problem 2}}$.

$$(1.13') \text{ NS} \begin{cases} \nabla \cdot (\rho \vec{u}) + \rho_t = 0 \\ \rho \vec{u}_t + \rho \vec{u} \cdot \nabla \vec{u} + \underset{\langle p_x, p_y, p_z \rangle}{\underset{\uparrow}{\nabla p}} = \rho \vec{F} \end{cases}$$

notation:

$\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$, $\nabla f = \langle f_x, f_y, f_z \rangle$, $\nabla \vec{u} = \langle \nabla u_1, \nabla u_2, \nabla u_3 \rangle$, $\vec{v} \cdot \langle \vec{z}_1, \vec{z}_2, \vec{z}_3 \rangle = \langle \vec{v} \cdot \vec{z}_1, \vec{v} \cdot \vec{z}_2, \vec{v} \cdot \vec{z}_3 \rangle$

$$\text{div} \vec{F} = \nabla \cdot \vec{F} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle M, N, P \rangle := M_x + N_y + P_z$$

$\sim$ so $\nabla \cdot (\rho \vec{u}) = \rho \,\text{div}\, \vec{u} + \nabla \rho \cdot \vec{u}$.

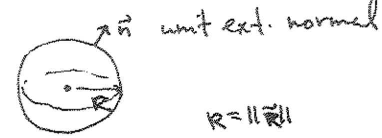
I will post a careful sol'n to old HW problem later this afternoon. That should help.

(4) Consider the vector field in $\mathbb{R}^3$ (singular at the origin)

$$\vec{E}(x,y,z) = \frac{c\,\vec{r}}{\|\vec{r}\|^3} \qquad \begin{array}{l} c \text{ const.} \\ \vec{r} = \langle x,y,z \rangle . \end{array}$$

with appropriate normalizing constants c this is the famous inverse square law which rules classical gravitational and electric force (and acceleration) fields. As Newton deduced, this was the only centripetal acceleration field consistent with Kepler's laws of gravitational motion. (This is one of my favorite mathematical derivations, see my Math 2210 notes, Spring 2010, February 3.)

4a) By computing the constant value of $\vec{E}\cdot\vec{n}$ of the sphere of radius R deduce that the flux integral



$$\iint_{\|\vec{x}\|=R} \vec{E}\cdot\vec{n} = 4\pi c$$

independently of R

4b) Compute $\operatorname{div}(\vec{E}) = \vec{\nabla}\cdot\vec{E}$, and show it is zero $\forall\, \vec{x} \neq \vec{0}$.

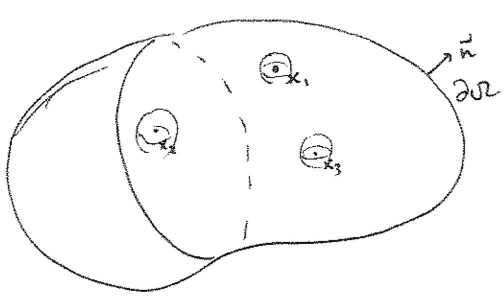
4c) Consider a sum of translations & scalings of the vector field $\vec{E}$,

$$\vec{F} = \frac{c_1(\vec{x}-\vec{x}_1)}{\|\vec{x}-\vec{x}_1\|^3} + \frac{c_2(\vec{x}-\vec{x}_2)}{\|\vec{x}-\vec{x}_2\|^3} + \dots + c_k\frac{(\vec{x}-\vec{x}_k)}{\|\vec{x}-\vec{x}_k\|^3}.$$

Use the divergence theorem, linearity of divergence and flux integral's wrt vector fields, to show that if Ω is any 3-d domain (with p.w. smooth $\partial\Omega$, so div thm applies), then

$$\iint_{\partial\Omega} \vec{F}\cdot\vec{n}\, dS = 4\pi\,(c_1+c_2+\dots+c_k)$$

Hint:



These considerations will lead us to $\boxed{\text{Maxwell's Equations}}$