Math 5440
Monday 9/27

Our plan! We'll finish discussing the harmonic function example today, and maybe have time to recall about where potential functions come from. (If not, you can read that portion of today's notes on your own.)

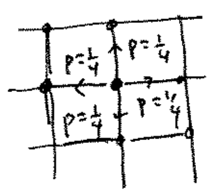
On Wednesday we'll derive the heat equation (easiest derivation by far!) as a model for how heat (or any of a lot of other quantities) diffuses over time. We'll also discuss uniqueness and estimates via energy (integral) and maximum principle methods. This in §3.13 ~ pretty readable if you want to start that HW.

On Friday we'll summarize the key points in Chapters 1-3, vector calculus, and derivations, to prepare for Monday's midterm exam. I will try my hardest to have all your HW graded and returned by Friday.

- There is a beautiful connection between the Friday example (and its generalization to any domain in $\mathbb{R}^n$), and probability theory:

there is the notion of a <u>random walk</u> in $\mathbb{R}^2$ (or $\mathbb{R}^n$), which is the <u>continuous limit</u> of a discrete random walk on a rectangular graph, where at each discrete time unit you have equal probability of moving in each possible coordinate direction to get your subsequent location.

$\mathbb{Z}^2 \subset \mathbb{R}^2$



(this is a discrete dynamical system).

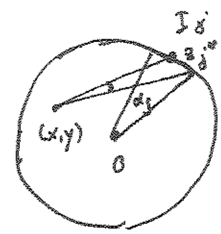
<u>continuous limit</u> means you take an appropriately scaled limit as the grid size $\to 0$.

<u>amazing fact</u>
the $u_I(x_0,y_0)$ function is the probability that a random walk starting at (x_0,y_0) first hits the boundary on the arc I. Notice that $u_{\partial D}(x_0,y_0) = 1$ (since the unique harmonic fcn with boundary values $= 1$ is the function 1; $u_I(x_0,y_0)$ is positive for $I \neq \emptyset$; additive with respect to disjoint unions of intervals (harmonic function superposition), so for fixed (x_0,y_0), this function defines a "probability measure" on the unit circle. Works for arbitrary bounded domains of sufficient regularity.

google something like "survey of harmonic measure and escape probability"
I found www.sandbergs.org/oskar/mag.pdf
Great project direction, if you're interested.

Details of the Riemann sum limit hinted at on page 5 of Friday's notes

$u_{I_j}(x,y)$:

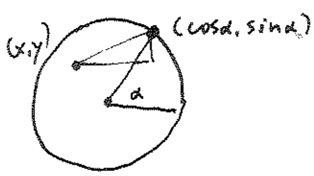


$$\sum_{j=1}^{n} f(z_j^*)\,\mathbb{1}_{I_j}(x,y) = \sum_{j=1}^{n} f(z_j^*)\,\frac{1}{\pi}\left(\beta_{j+1}(x,y)-\beta_j(x,y)-\frac{\alpha_j}{2}\right)$$

$$= \sum_{j=1}^{n} f(z_j^*)\,\frac{1}{\pi}\left[\frac{\beta_{j+1}(x,y)-\beta_j(x,y)}{\alpha_j}\right]\alpha_j \;-\; \frac{1}{2\pi}\sum_{j=1}^{n} f(z_j^*)\,\alpha_j$$

$$\Big\downarrow \; \|P\|\to 0$$

$$\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\partial}{\partial\alpha}\beta(x,y,\alpha)\right) f(\alpha)\,d\alpha \;-\; \frac{1}{2\pi}\int_0^{2} f(\alpha)\,d\alpha$$



$$\tan\beta = \frac{\sin\alpha - y}{\cos\alpha - x}$$

$$\Rightarrow \sec^2\beta\,\frac{\partial\beta}{\partial\alpha} = \frac{\cos\alpha(\cos\alpha - x) - (\sin\alpha - y)(-\sin\alpha)}{(\cos\alpha - x)^2}$$

$$\frac{(\cos\alpha - x)^2 + (\sin\alpha - y)^2}{\cancel{(\cos\alpha - x)^2}}\,\frac{\partial\beta}{\partial\alpha} = \frac{1 - x\cos\alpha - y\sin\alpha}{\cancel{(\cos\alpha - x)^2}}$$

$$\frac{\partial\beta}{\partial\alpha} = \frac{1 - \langle x,y\rangle\cdot\langle\cos\alpha,\sin\alpha\rangle}{\|\langle x,y\rangle - \langle\cos\alpha,\sin\alpha\rangle\|^2}$$

$\Rightarrow$ the limit function $u(x,y)$ solving Dirichlet problem is

$$u(x,y) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{2 - \cancel{2\langle x,y\rangle\cdot\langle\cos\alpha,\sin\alpha\rangle} - \|\langle x,y\rangle - \langle\cos\alpha,\sin\alpha\rangle\|^2}{\|\langle x,y\rangle - \langle\cos\alpha,\sin\alpha\rangle\|^2}\right] f(\alpha)\,d\alpha$$

$$\|\langle x,y\rangle - \langle\cos\alpha,\sin\alpha\rangle\|^2 = \|\langle x,y\rangle\|^2 - \cancel{2\langle x,y\rangle\cdot\langle\cos\alpha,\sin\alpha\rangle} + 1$$

$$\boxed{u(x,y) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1 - \|\langle x,y\rangle\|^2}{\|\langle x,y\rangle - \langle\cos\alpha,\sin\alpha\rangle\|^2}\,f(\alpha)\,d\alpha}$$

Poisson Integral formula 24.12 page 103.

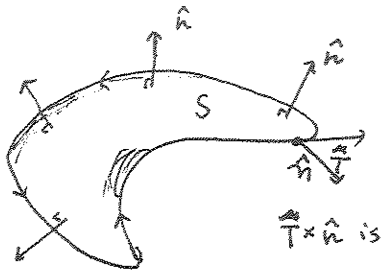
Notice this is an integral superposition of harmonic functions : $v_\alpha(x,y) = \dfrac{1 - \|\langle x,y\rangle\|^2}{\|\langle x,y\rangle - \langle\cos\alpha,\sin\alpha\rangle\|^2}$

(well, it's a computation to check $\Delta v_\alpha = 0$),

then if you can justify passing Δ through the integral sign, done!

4) Stokes' Theorem

last class exercise hw this week



$$\boxed{\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS = \oint_{\partial S} \vec{F}\cdot\vec{T}\,ds}$$

$\hat{T}\times\hat{n}$ is outer conormal $\hat{n}$

This is the last classical vector integration theorem to check! (And you probably used it to derive wave eqtn for transverse membrane oscillation.)

→ we did $\mathbb{R}^n$ FTC

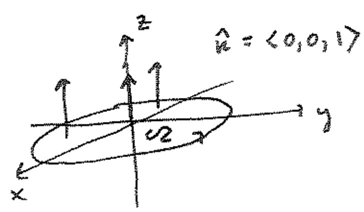
⇒ $\mathbb{R}^n$ div thm

$\mathbb{R}^2$ div thm ⇒ $\mathbb{R}^2$ Green.

(a) If $\vec{F} = \langle M(x,y), N(x,y)\rangle$ is an $\mathbb{R}^2$ vector field, consider $\mathbb{R}^2$ has the $z=0$ plane in $\mathbb{R}^3$, extend $\vec{F}$ as $\langle M, N, 0\rangle$ in $\mathbb{R}^3$

Let $S = \Omega$ a domain in $\mathbb{R}^2$

and Check that Green's Theorem is a special case of Stokes' Theorem.

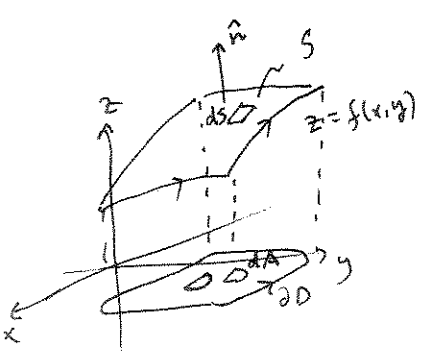


(b) In fact, Stokes' Theorem follows from Green's Theorem, via decomposition of S into pieces which are graphs above one of the coordinate planes

You apply Green's Theorem on each graphical piece, for the appropriate 2 variables xy, yz, zx

Notice that the line integrals on edges which glue together cancel out, leaving only the line integral around ∂S.

- You will check that for a graph $z = f(x,y)$, Stokes' theorem on the graph reduces to Greens' theorem on the x-y domain.



$$\hat{n} = \frac{\langle -u_x, -u_y, 1\rangle}{\sqrt{1+u_x^2+u_y^2}}$$

$$dS = \sqrt{1+u_x^2+u_y^2}\,dA$$

If $\langle x(t), y(t)\rangle$ parameterizes part of ∂D, then $\langle x(t), y(t), f(x(t), y(t))\rangle$ parameterizes the graph boundary ∂S above ∂D.

$\vec{F} = \langle M(x,y,z), N, P\rangle$

and once you compute $\vec{\nabla}\times\vec{F}$ it will be evaluated at the parameterized points $(x, y, f(x,y))$

Verify Stokes' for this graph by invoking Green's Theorem in D.

Potential functions

φ is called a potential function for a vector field $\vec{F}$ if $-\nabla\varphi = \vec{F}$. The reason for this name is that if a particle of mass m is accelerated by (force) $\vec{F}$ according to Newton's 2^{nd} law

$$\text{N2} \qquad m\vec{x}'' = \vec{F}(\vec{x}(t))$$

Then for total energy

$$TE := KE + PE := \underbrace{\tfrac{1}{2}m\|\vec{x}'(t)\|^2}_{KE} + \underbrace{\varphi(\vec{x}(t))}_{PE}$$

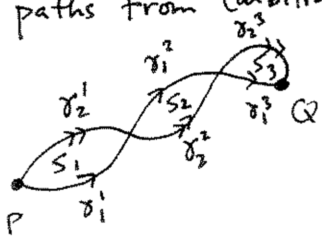
TE is identically constant. In this case we call the vector field conservative because TE is "conserved". (We also call it a gradient field.)

a) Check N2 $\Rightarrow$ TE is constant for particle motion above.

b) Assuming it's possible for different particles to pass a fixed point with arbitrary velocities, prove the converse to (a); namely TE constant for all paths $\Rightarrow$ N2. (We used this principle often in 2280 to derive 2^{nd} order dynamical system differential equations.)

c) Any gradient field (C^1) is curl free: check $\vec{\nabla}\times\vec{\nabla}\varphi = 0$

d) The converse to (c) is true on suitable domains ("simply connected"). The rough idea for this fact is Stokes' theorem, although that's not exactly how the rigorous proof goes.... If $\vec{\nabla}\times\vec{F} \equiv \vec{0}$ in Ω, and if any two paths from (arbitrary) points P to Q, say γ_1 and γ_2, can be decomposed so that $\gamma_1 - \underbrace{\gamma_2}_{\gamma_2 \text{ in the other direction}}$



$\partial S_1 = \gamma_1^1 - \gamma_2^1$
$\partial S_2 = \gamma_2^2 - \gamma_1^2$
$\partial S_3 = \gamma_1^3 - \gamma_2^3$

is the boundary of a union of surfaces (or generalized self-intersecting surfaces), with appropriate orientations

then
$$\int_{\gamma_1-\gamma_2} \vec{F}\cdot d\vec{x} \underset{\text{Stokes'}}{=} \sum \iint_{S_i} (\vec{\nabla}\times\vec{F})\cdot\vec{n} = \vec{0}$$

So line integrals are path independent.

Then $\varphi(\vec{x}) := -\int_{\gamma} \vec{F}\cdot d\vec{x}$, γ any path from P to $\vec{x}$, defines a potential function s.t. $\nabla\varphi = -\vec{F}$.

for (d) read as much or as little (say on wiki) about the precise def'n of simply connected, which makes precise the rough idea that closed curves bound a surface in the domain $\sim$ in $\mathbb{R}^2$ this means roughly that the domain have no holes.

Examples of potential functions:

$$\vec{E} = + \frac{\vec{x}-\vec{x}_0}{\|\vec{x}-\vec{x}_0\|^3}$$

has potential function $\varphi = \frac{1}{\|\vec{x}-\vec{x}_0\|}$ in $\mathbb{R}^3$; $-\nabla\varphi = \vec{E}$ (computation).

Thus any electric field which is a finite superposition of static point charges is (or gravity) a potential field way from the singularities.

Any magnetic field which is a finite superposition of static dipoles ($\vec{B} = \nabla\vec{E}\cdot\vec{w}$, $\vec{w}$ is called the moment; $\vec{B}$ is a rescaled limit of opposite pt-charges, as distance $\to 0$) has potential $\nabla\varphi\cdot\vec{w}$, so is also a potential field.

Since $\operatorname{div}\vec{E} = 0$ (you did this), also $\operatorname{div}\vec{B} = 0$, and so

$\Delta\varphi = 0$ in these cases.

For the continuous charge (or magnetic distributions), $\vec{E}$ & $\vec{B}$ are still potential fields

$$\boxed{\begin{aligned} \operatorname{div}\vec{E} &= 4\pi\rho \\ \vec{\nabla}\times\vec{E} &= \vec{0} \\ \operatorname{div}\vec{B} &= 0 \\ \vec{\nabla}\times\vec{B} &= \vec{0} \end{aligned}}$$

static Maxwell eqtns.

$\left(\begin{matrix} \text{so } \Delta\varphi = F \\ F = 4\pi\rho. \end{matrix} \right)$

$\left(\int_{\partial E} \vec{E}\cdot\vec{n} = 4\pi\int_{\Omega} \rho \, dV \right.$ ← limit of old HW

$\qquad || \\ \int_E \operatorname{div}\vec{E}\, dV$

(I don't know as simple a derivation for the dynamic Maxwell eqtns.)