Math 5440
Friday 9/24

§3.12 & example

Maximum & comparison principles for Laplacian (and other elliptic operators)

Maximum Principle

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain
with closure $\bar{\Omega} = \Omega \cup \partial\Omega$.
Let $w \in C^2(\Omega) \cap C(\bar{\Omega})$.

graph(w)

Let $\Delta w \geq 0 \quad \forall x \in \Omega$
Then the maximum value occurs on $\partial\Omega$

proof: Case I : Assume the strict inequality $\Delta w > 0 \quad \forall x \in \Omega$.
From analysis, w attains its max value on the compact domain $\bar{\Omega}$.
It may occur on $\partial\Omega$, in Ω, or both.
We show in case I that the max cannot occur in Ω:

If $w(\vec{x}_0)$ is a max value with $\vec{x}_0 \in \Omega$
Then $\nabla w(\vec{x}_0) = \vec{0}$
and each second directional derivative $u_{\xi\xi} \leq 0$ (concave down)
so each $\frac{\partial^2}{\partial (x^k)^2} u \leq 0$
so $\Delta u(\vec{x}_0) \leq 0$ violates $\Delta u > 0$ in Ω.
Thus in this case, max value can only occur on $\partial\Omega$ ∎

Case II If only $\Delta w \geq 0$ holds $\forall x \in \Omega$

Let $w_\varepsilon(\vec{x}) = w(\vec{x}) + \varepsilon \|\vec{x}\|^2 \qquad \varepsilon > 0$
$\Delta w_\varepsilon = \Delta w + 2n\varepsilon \qquad (\Omega \subset \mathbb{R}^n)$
so $\Delta w_\varepsilon > 0$ in Ω
so by case I

$$\max_{x \in \bar{\Omega}} w(x) \leq \max_{x \in \bar{\Omega}} w_\varepsilon(x) = \max_{x \in \partial\Omega} w_\varepsilon(x) \leq \left(\max_{x \in \partial\Omega} w(x)\right) + \varepsilon R^2$$

where $\Omega \subset B_R(0)$

Let $\varepsilon \to 0$ ∎

Remark By our orthogonal COV discussion we see
this same proof would work if Δ was replaced by
$$L = a^{ij}\partial_{x^i}\partial_{x^j} + b^k\partial_{x^k}$$
as long as all the eigenvalues of the symmetric matrix $[a^{ij}]$
are ≥ 0, and at least one of them is positive.

Comparison Principle

Let $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$
with Ω as before.
If $u \leq v$ on $\partial\Omega$
$\Delta u \geq \Delta v$ in Ω
Then $u \leq v$ in $\bar{\Omega}$

physical interpretation $n=1$, $n=2$.
$-\Delta u$ = body forces.
$-\Delta u \leq -\Delta v$ says the body forces for sol'n u are less than for v.

proof: Apply the maximum principle to $w = u - v$: $\Delta w \geq 0$ in Ω, $w \leq 0$ on $\partial\Omega$
so $w \leq 0$ in $\bar{\Omega}$ so $u \leq v$ in $\bar{\Omega}$ ∎

Corollary: Solutions to the Dirichlet problem ($u \in C^2(\Omega) \cap C(\bar{\Omega})$, Ω bd.)
are unique:

$$DP \begin{cases} \Delta u = F & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

∎

proof: Comparison principle $\Rightarrow$ $u \leq v$ and $v \leq u$

Better corollary: uniqueness with estimates

Let Ω as above, $\Omega \subset B_R(\vec{x}_0)$
Suppose $\exists\, \varepsilon_1, \varepsilon_2 > 0$ s.t.
$|\Delta u - \Delta v| \leq \varepsilon_1$ in Ω
$|u - v| \leq \varepsilon_2$ on $\partial\Omega$

Then $|u(x) - v(x)| \leq \varepsilon_2 + \frac{\varepsilon_1}{2n} R^2 \quad \forall x \in \bar{\Omega}$.

proof: compare
u to $v + \varepsilon_2 + \frac{\varepsilon_1}{2n}(R^2 - \|\vec{x} - \vec{x}_0\|^2) \Rightarrow u \leq v + \varepsilon_2 + \frac{\varepsilon_1}{2n} R^2$ in $\bar{\Omega}$
v to $u + \varepsilon_2 + \frac{\varepsilon_1}{2n}(R^2 - \|\vec{x} - \vec{x}_0\|^2) \Rightarrow v \leq u + \varepsilon_2 + \frac{\varepsilon_1}{2n} R^2$ ''

∎

Remark There is a version of Max. princ. & Comp. principle for Neumann data too. Here's Max. princ. version

Strong Maximum Principle (Neumann data): Let $\Omega \subset \mathbb{R}^n$ be bounded, with $\partial\Omega$ smooth. Let $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$.
If $\Delta w \geq 0$ in Ω, so that max value of Ω occurs on the boundary $\partial\Omega$ as above, then at any $x_0 \in \partial\Omega$ with $w(x_0) = \underset{x \in \bar{\Omega}}{\text{maximum}}\, w(x)$, $\frac{\partial w}{\partial n}(\vec{x}_0) > 0$ OR w is identically constant.

There is also a strong version of the $\begin{cases}\text{Maximum principle for Dirichlet data:}\\ \text{Comparison}\end{cases}$

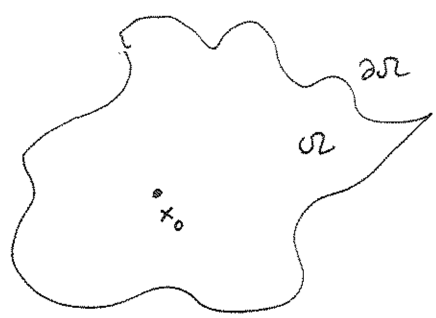
<u>Strong Comparison Principle</u>

Let Ω bd, open, with boundary $\partial\Omega$.
Let $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$
If $u \leq v$ on $\partial\Omega$
$\Delta u \geq \Delta v$ in Ω
Then $u \leq v$ in $\bar{\Omega}$.

<u>Furthermore</u> if $u < v$ anywhere on $\partial\Omega$
or if $\Delta u > \Delta v$ anywhere in Ω
then $u < v$ everywhere in Ω

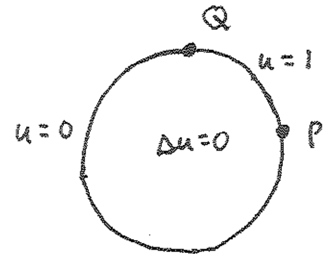
What does the strong comparison principle say about the domain of dependence for the point $x_0 \in \Omega$, for the Dirichlet problem

$$\begin{cases} \Delta u = F & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$



<u>Example</u> Consider a special Dirichlet problem for harmonic functions on the unit disk D in $\mathbb{R}^2$

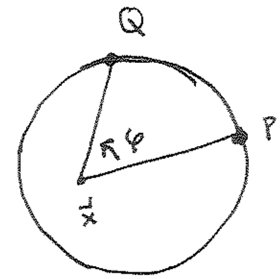
$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = \mathbb{1}_{PQ} & \text{on } \partial D \end{cases}$$



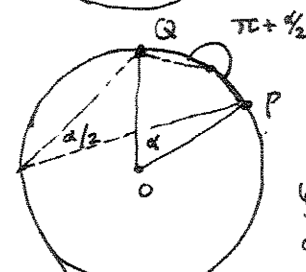
"identifier" or "indicator" or "characteristic fcn"

$$\mathbb{1}_{PQ}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \text{arc}(PQ) \quad \text{(counterclockwise)} \\ 0 & \text{if } (x,y) \notin \text{arc}(PQ) \end{cases}$$

Here is a solution! Let $\varphi(\underbrace{x,y}_{\vec{x}}) = \angle\, P-\vec{x},\, Q-\vec{x}$



- φ is harmonic because it's a linear combination (+ const.) of the polar coordinate angle θ, with poles at P & Q. (Such θ functions are harmonic – see HW!)



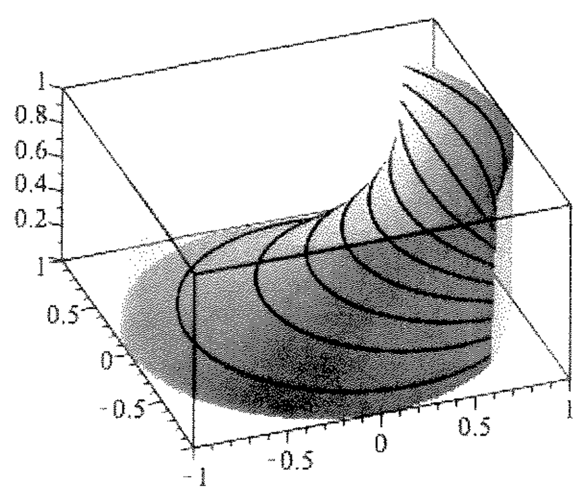
Geometry! for the boundary values of φ!

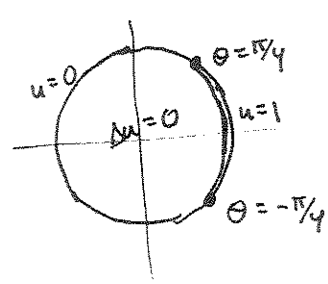
$\varphi = \frac{\alpha}{2}$ on the complementary arc to PQ.

$\varphi = \pi + \frac{\alpha}{2}$ on PQ

- So $u(x,y) = \frac{1}{\pi}\left(\varphi(x,y) - \frac{\alpha}{2}\right)$ solves this special Dirichlet Problem

- Even though u is <u>not</u> continuous on $\bar{D}$, it is the only bounded harmonic function which solves this Dirichlet problem. (see HW!)

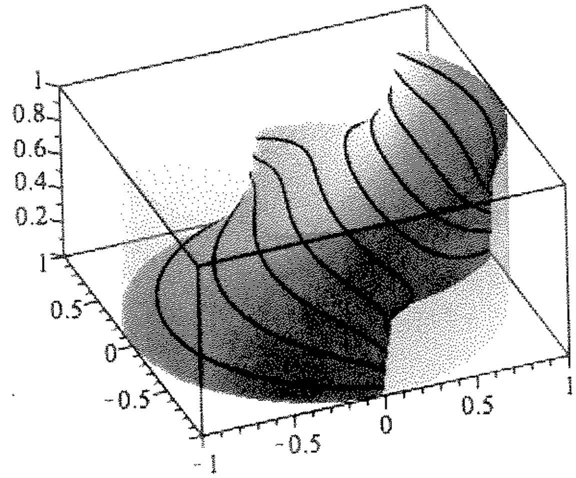


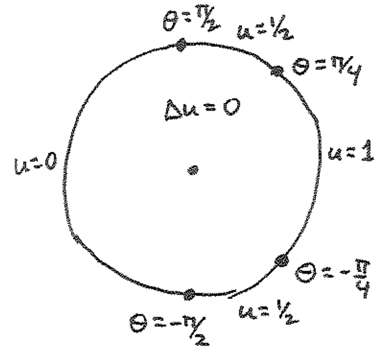


notice how everything gets smoothed out in the interior! No propagation of singularities

Math 5440 is a course about using superposition to solve linear PDE's.

Watch this!

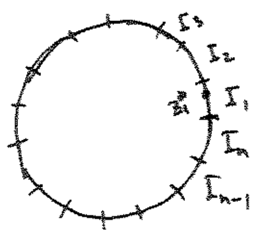




linear combination

of 3 solutions!

Now consider continuous $f(\theta)$ on the unit circle.
Partition circle.



Pick $z_j^* \in I_j$

consider the harmonic function (because linear combo of harmonic)

$$u(x,y) = \sum_{j=1}^{n} f(z_j^*)\, u_{I_j}(x,y)$$

f at a sample point

the harmonic solution for the indicator Dirichlet problem on interval I_j

As the norm of the partition $\to 0$

$$\sum_{j=1}^{n} f(z_j^*)\, \mathbb{1}_{I_j}(x,y) \to f \text{ uniformly. on } \partial D$$

<u>Maximum principle</u> $\Rightarrow$ $\sum_{j=1}^{n} f(z_j^*)\, u_{I_j}(x,y)$ converges uniformly to a limit function in D

unif.

for harmonic fcns, uniform conv $\Rightarrow$ all derivs also converge (uniformly away from ∂D)

u, where u solves $\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \partial D \end{cases}$!

We'll solve this with a different superposition method later in course (uses Fourier series)

Homework

1 The 2 best harmonic functions, $\ln r$ and θ.

Consider polar coordinates in $\mathbb{R}^2$, (r, θ), with $\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}$. $r = \sqrt{x^2+y^2}$

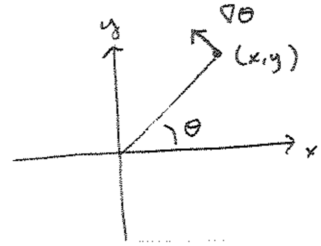
$$\theta = \begin{cases} \arctan y/x & \mathrm{I}, \mathrm{IV} \\ \arccos \frac{x}{\sqrt{x^2+y^2}} & \mathrm{I}, \mathrm{II} \\ \text{etc.} \end{cases}$$

II	I
III	IV

a) Show $\ln r = \ln\sqrt{x^2+y^2}$ is harmonic
by showing $(\partial_x^2 + \partial_y^2)(\ln r) = 0$

b) Although the formula for θ is messy and involves switching between inverse trig functions depending on quadrant the formula for $\nabla\theta$ is easy:

$$\langle \theta_x, \theta_y \rangle = \frac{1}{x^2+y^2} \langle -y, x \rangle$$



Derive this formula <u>without</u> resorting to inverse trig fcns.
<u>Hint</u>: Recall that the gradient of a function points in the direction of maximum (unit) directional derivative, and that its magnitude equals this directional derivative.

c) Using the formula for $\nabla\theta$ in (b), show θ is harmonic

d) Show that

$$\Delta u = u_{xx} + u_{yy} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r^2} u_{\theta\theta}$$

via chain rule

e) Use (d) to recheck that
θ and $\ln r$ are harmonic functions.

<u>Remark</u> The complex analytic function $\log z = \ln|z| + i\theta$; $\ln r$ and θ are harmonic conjugates
(θ also called $\arg(z)$)

<u>Remark</u> If $u(\vec{x})$ is harmonic then
$u(\vec{x} - \vec{x}_0)$ is too (translations don't affect harmonicity)

So angles from any pole P is a harmonic fcn, (and we can measure the angle from any fixed direction and in either the clockwise or counterclockwise directions.)
as is $\ln\|\vec{x} - \vec{x}_0\|$.

Homework

2 Extended maximum principle in $\mathbb{R}^2$

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with closure $\bar{\Omega} = \Omega \cup \partial\Omega$

Let $w \in C^2(\Omega)$ and continuous except at a finite number of boundary points $P_1, P_2, .. P_k$.

Let w be bounded, $|w(x,y)| \le M \quad \forall (x,y) \in \bar{\Omega}$

Let

$$\Delta w \geq 0 \text{ in } \Omega.$$

Then if $w \le 0$ on $\partial\Omega$, $w \le 0$ on $\bar{\Omega}$

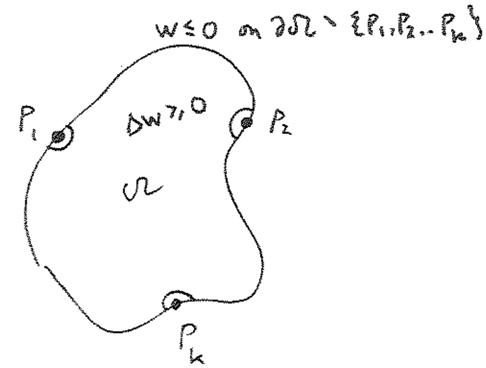
we mean $w \le 0$ on $\partial\Omega$, except at $P_1, P_2, .. P_k$ where any def'n of w is irrelevant.

Prove this theorem.

Hint: Consider

$$w + \varepsilon \sum_{j=1}^{k} \ln\left(\frac{\|\vec{x} - \vec{P}_j\|}{R}\right)$$

for fixed R (s.t. $\Omega \subset B_R(P_j) \ \forall j$), and for subdomains Ω_ε obtained from Ω by deleting small disks (radius depending on ε) about each P_j. Apply Max. principle to these domains, let $\varepsilon \to 0$.



3 3.13.2 (Heat equation)
3.13.4

more to follow.