Math 5440
Monday Sept 20

(there's one more exercise for Friday – see page 4)

- Finish Friday notes on 2^{nd} order linear PDE classification

§3.10–3.11 : Laplace eqtn.

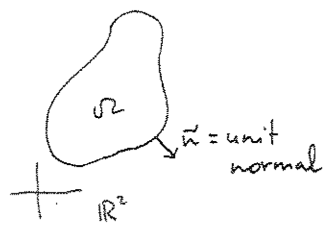
The two most natural BVP's for Laplace eqtn are

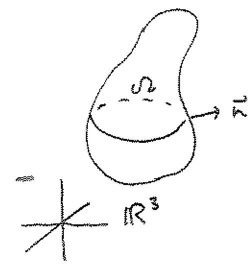
$$\text{DP}\begin{cases}\Delta u = F & \text{in } \Omega \subset \mathbb{R}^n\\ u = f & \text{on } \partial\Omega\end{cases}$$

Dirichlet problem (fixed boundary values.

$$\text{NP}\begin{cases}\Delta u = F & \text{in } \Omega \subset \mathbb{R}^n\\ \nabla u \cdot \vec{n} = g & \text{on } \partial\Omega\end{cases}$$

prescribed normal derivative
Neumann problem.





usually Ω is a bounded, piecewise smooth domain.

Laplace's eqtn arises as steady-state soltns for the wave equation & the heat equation.

As for wave equation analog, uniqueness for DP, NP follows from the result that the homogeneous problem has only the identically zero soltn. (See HW.)

Theorem Let Ω be bounded and piecewise smooth then the only soltns to either

$$\begin{cases}\Delta u = 0 & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega\end{cases} \quad \text{or} \quad \begin{cases}\Delta u = 0 & \text{in } \Omega\\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega\end{cases} \qquad u \in C^2(\Omega) \cap C^1(\bar{\Omega})$$

actually only need piecewise C^1

are $u \equiv 0$, (or u = constant for NP)

pf: Recall $\nabla \cdot (h\vec{v}) = \nabla h \cdot \vec{v} + h(\nabla \cdot \vec{v})$

divergence; scalar fcn; vector field

$$\frac{\partial}{\partial x^k}(hv^k) = \frac{\partial h}{\partial x^k}v^k + h\frac{\partial v^k}{\partial x^k}$$
(summed over k)

thus, on any domain for which div thm applies,

$$\int_{\partial\Omega} (u\nabla u)\cdot\vec{n} = \int_{\Omega} \operatorname{div}(u\nabla u)\, dV = \int_{\Omega} \underbrace{\nabla u \cdot \nabla u}_{|\nabla u|^2} + u\,\Delta u \; dV$$

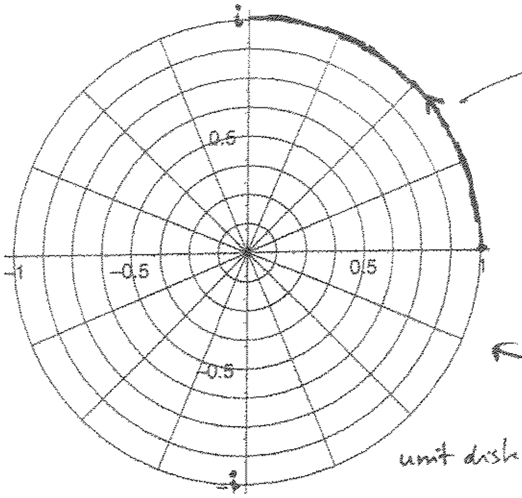
So if $u = 0$ or $\nabla u \cdot n = 0$
at all points of $\partial\Omega$
and if $\Delta u \equiv 0$ in Ω, deduce $\int_{\Omega} |\nabla u|^2 dV = 0. \implies |\nabla u| \equiv 0$ in $\Omega \Rightarrow u = \text{const}$

for DP deduce $u \equiv 0$ ∎

(notes from Math 4200) (Fall 2007 Dec. 3)

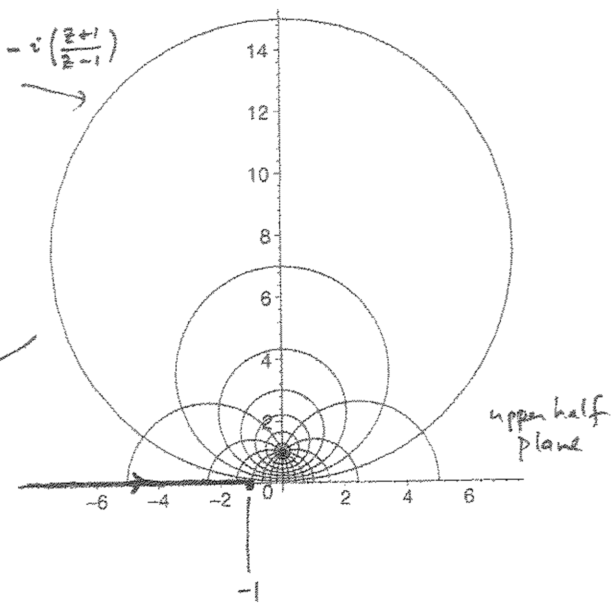
- Applications of conformal mapping to harmonic functions. Recall, harmonic $\circ$ analytic = harmonic. (Why?)

Use Friday FLT as example



$f(z) = -i\left(\frac{z+1}{z-1}\right)$

$g(w) = \frac{w-i}{w+i}$



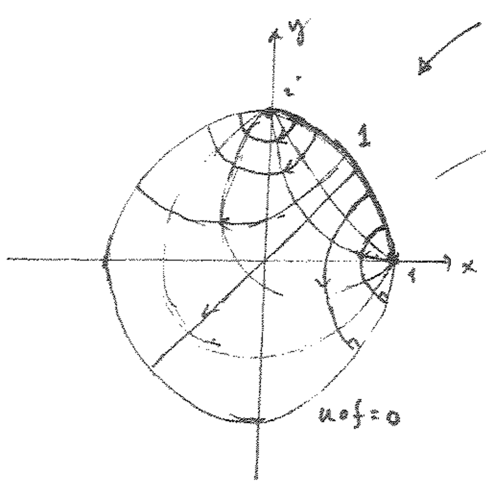
$0 \longleftrightarrow i$
$1 \longleftrightarrow \infty$
$-1 \longleftrightarrow 0$
$-i \longleftrightarrow 1$
$i \longleftrightarrow -1$

$$u(x,y) = \arccos \frac{x+1}{\sqrt{(x+1)^2+y^2}}$$

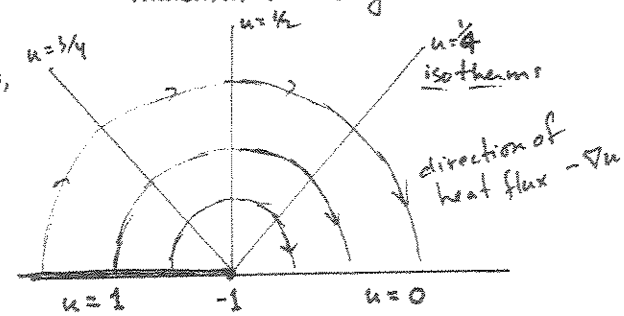
Consider $u(w) = \frac{\arg(w-(-1))}{\pi} = \frac{\arg(w+1)}{\pi}$

Is harmonic, with indicated boundary values:

Thus, $u \circ f(z)$ solves harmonic fcn with corresponding boundary temperatures, isotherms & temp grads



$f(z)$



other applications: fluid mechanics, airfoils.

Maple visualization of previous page

harmonic function examples, using complex analytic functions.

> $with(plots):$

>

> $f := (x, y) \to \left(\mathrm{Re}\left(\frac{((x + I\cdot y) + 1)}{I\cdot(x + I\cdot y) - I}\right), \Im\left(\frac{((x + I\cdot y) + 1)}{I\cdot(x + I\cdot y) - I}\right)\right);$

#conformal transformation of unit disk to upper half plane

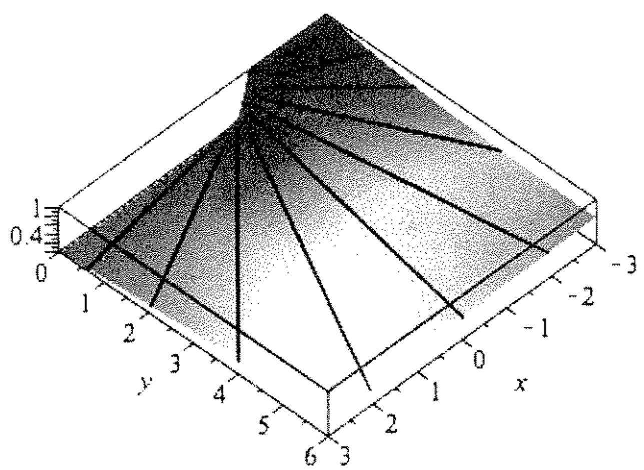
$$f := (x, y) \to \left(\Re\left(\frac{x + \mathrm{I}\,y + 1}{\mathrm{I}\,(x + \mathrm{I}\,y) - \mathrm{I}}\right), \Im\left(\frac{x + \mathrm{I}\,y + 1}{\mathrm{I}\,(x + \mathrm{I}\,y) - \mathrm{I}}\right)\right) \quad (1)$$

>

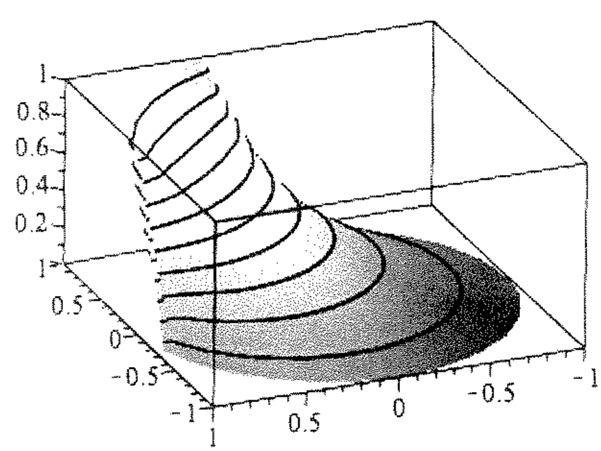
> $u := (x, y) \to \frac{1}{\mathrm{Pi}}\cdot \mathrm{Im}(\log(x + I\cdot y + 1));$

$$u := (x, y) \to \frac{\Im(\log(x + \mathrm{I}\,y + 1))}{\pi} \quad (2)$$

> $plot3d(u(x, y), x = -3..3, y = 0..6, axes = boxed, scaling = constrained);$



> $plot3d([r\cdot\cos(\mathrm{theta}), r\cdot\sin(\mathrm{theta}), u(f(r\cdot\cos(\mathrm{theta}), r\cdot\sin(\mathrm{theta})))], r = 0..(1), \mathrm{theta} = 0..2\cdot\mathrm{Pi}, scaling = constrained, axes = boxed);$

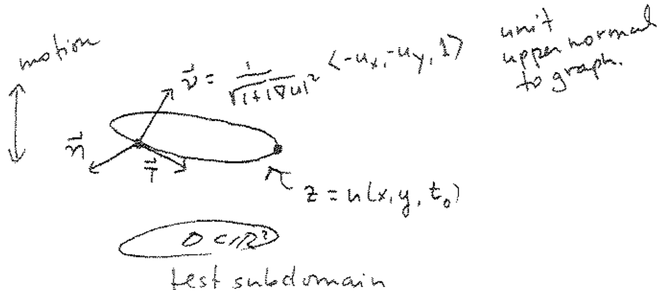


Class exercise 1 For Friday Sept 24

2 spacedimension wave eqtn as linearized model
for transverse vibration of membrane (e.g. drumhead.).

Assume $T = T_0$ = constant surface tension, $\rho = \rho_0$ = density mass/length, vibration is vertical

this is force/length magnitude along a vertical cut. Actual force is $T_0 \vec{n}$ where $\vec{n}$ is unit conormal vector to surface



$\vec{n} = \vec{T} \times \vec{\nu}$

oriented unit tangent to boundary curve.

So, assume surface tension force on the graph of the test subdomain D ($\to S$)

is $\oint_{\partial S} T_0 \vec{n}\, ds$

so its vertical component is $\oint_{\partial S} T_0 \vec{n}\cdot\vec{e}_3\, ds$

Use geometry & Stoke's Theorem, and Newton's 2nd law, and linearize, to get

$$\boxed{u_{tt} - c^2(u_{xx}+u_{yy}) = F \quad \text{with } c = \sqrt{\frac{T_0}{\rho_0}}}$$

(F = vertical body force/mass)