Math 5440
Friday 9/17

2.8 (skip 2.9), part of 3.10

$\boxed{\text{Change of variables day}}$

if you change variables $\vec{x} = \vec{x}(\vec{\xi})$
and if $u(\vec{x})$ satisfies a P.D.E., $\bar{u}(\xi) = u(\vec{x}(\vec{\xi}))$ satisfies a transformed PDE.
It might be easier to solve!

Example 0 the wave equation $u_{tt} - c^2 u_{xx} = 0$

transforms to $\bar{u}_{tt} - \bar{u}_{\tilde{x}\tilde{x}} = 0$ if $\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} t \\ c\tilde{x} \end{bmatrix}$

since $\bar{u}_{\tilde{x}} = u_x \frac{\partial x}{\partial \tilde{x}} = c u_x$

$\Rightarrow \bar{u}_{\tilde{x}\tilde{x}} = c u_{xx} \frac{\partial x}{\partial \tilde{x}} = c^2 u_{xx}$ ■

$(\bar{u}_{tt} = u_{tt})$

(so taking $c = 1$ isn't unreasonable)
as long as you let ℓ vary

Example 1 find the general soln to

$$\boxed{u_{xx} + 2u_{xy} + u_{yy} = 0}$$

by trying to factor $L = \partial_x^2 + 2\partial_x\partial_y + \partial_y^2$.

$L = (\partial_x + \partial_y) \circ (\partial_x + \partial_y)$

suggests $\frac{\partial}{\partial \xi} = \partial_x + \partial_y$, so that $L = \left(\frac{\partial}{\partial \xi}\right)^2$

$\frac{\partial}{\partial \eta} = \partial_y$ ← any directional deriv not // to $\partial_x + \partial_y$ would do.

chain rule: $\frac{\partial}{\partial \xi} = x_\xi \frac{\partial}{\partial x} + y_\xi \frac{\partial}{\partial y}$

$\frac{\partial}{\partial \eta} = x_\eta \frac{\partial}{\partial x} + y_\eta \frac{\partial}{\partial y}$

deduce $x_\xi = 1 \quad y_\xi = 1$
$x_\eta = 0 \quad y_\eta = 1$

$x = \xi$
$y = \xi + \eta$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$

$\begin{bmatrix} x \\ y-x \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$

$\boxed{\bar{u}_{\xi\xi} = 0}$

$\int d\xi$: $\bar{u}_\xi = q(\eta)$

$\int d\xi$: $\bar{u} = \xi q(\eta) + p(\eta)$

$\Rightarrow \boxed{u(x,y) = x\, q(y-x) + p(y-x)}$ p, q arbitrary fcns

Check!

Theorem : page 2 Wednesday notes.

the correct statement should have said that either $+L$ or $-L$ had one of the three canonical forms, depending on whether L was hyperbolic, parabolic, or elliptic.

Catalog of basic 2nd order linear pde operators:

1 space dim:

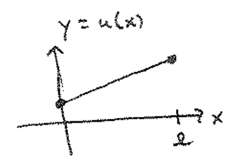
$L = \partial_t^2 - \partial_x^2$ wave

$L = \partial_t - \partial_x^2$ heat

steady state sol'ns to $Lu = f(x)$

$-u_{xx} = f$

$u_{xx} = 0$ homogeneous soltns $u(x) = ax + b$.



2 space dim

$L = \partial_t^2 - (\partial_x^2 + \partial_y^2)$ wave

$L = \partial_t - (\partial_x^2 + \partial_y^2)$ heat

steady state sol'ns to $Lu = f$

$-(u_{xx} + u_{yy}) = f$ $\qquad \Delta = \mathrm{div}(\nabla) = \partial_{xx} + \partial_{yy}$

$\Delta u = u_{xx} + u_{yy} = 0$ homogeneous soltns.

3 space dim

$L = \partial_t^2 - (\partial_x^2 + \partial_y^2 + \partial_z^2)$ wave

$L = \partial_t - (\partial_x^2 + \partial_y^2 + \partial_z^2)$ heat

steady state soltns to $Lu = f$

$-\Delta u = f$

$\Delta u = 0$ homogeneous.

in any space dim n, soltns to $\Delta u = 0$ are called <u>harmonic functions</u>

$n = 1$: boring affine fcns

$n \geq 2$: very interesting

$n = 2$: connection to complex analysis!

$$\partial_{xx} + \partial_{yy} = (\partial_x + i\partial_y) \circ (\partial_x - i\partial_y)$$

if $f(z) = f(x+iy)$ is <u>complex analytic</u>, then ($f = u + iv$)

$f_x + i f_y = 0$ (Cauchy Riemann-eqtns)

so $Re(f) = u$ and $Im(f) = v$ are <u>harmonic</u>

Classification of linear 2nd order PDE's in n variables

notation : vector components will be denoted with superscripts , not subscripts

$$\vec{x} = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}, \quad \xi = \begin{bmatrix} \xi^1 \\ \xi^2 \\ \vdots \\ \xi^n \end{bmatrix}$$

partial derivs will still be subscripts.

any time an index is repeated – once as a subscript, once as a superscript, it is assumed to be summed from 1 to n (summation convention)

Step 1

Let $u = u(\vec{x})$ solve a linear 2nd order PDE, $Lu = f$.

Then since $u_{x^i x^j} = u_{x^j x^i}$ we may write

$$Lu = L_2 u + L_1 u + L_0 u$$

summed over $i,j = 1..n$ → $L_2 = a^{ij}\partial_{x^i}\partial_{x^j}$ $\quad [a^{ij}]$ symmetric

summed $k = 1..n$ → $L_1 = b^k \partial_{x^k}$

$L_0 = cI$

the a^{ij}, b^k, c may be constants, or functions of $\vec{x}$

example $\partial_x^2 + 2\partial_x\partial_y + \partial_y^2 + 3\partial_x + 4\partial_y + 5$

$$= [\partial_x \ \partial_y]\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} + [3, 4]\begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} + 5$$

Step 2

COV $\quad \bar{u}(\xi) = u(\vec{x}(\xi))$

$u(\vec{x}) = \bar{u}(\xi(x))$

$u_{x^i} = \bar{u}_{\xi^k}\dfrac{\partial \xi^k}{\partial x^i}$ ← summed over k

$u_{x^i x^j} = \bar{u}_{\xi^k \xi^l}\dfrac{\partial \xi^k}{\partial x^i}\dfrac{\partial \xi^l}{\partial x^j} + \bar{u}_{\xi^k}\dfrac{\partial^2 \xi^k}{\partial x^i \partial x^j}$

disappears if $\xi(\vec{x})$ is a linear transformation

So, $\quad L_2 u = a^{ij}u_{x^i x^j} = \left(\dfrac{\partial \xi^k}{\partial x^i}a^{ij}\dfrac{\partial \xi^l}{\partial x^j}\right)\bar{u}_{\xi^k \xi^l} + \left(a^{ij}\xi^k_{x^i x^j}\right)\bar{u}_{\xi^k}$

$L_1 u = b^i u_{x^i} = (b^i \xi^k_{x^i})\bar{u}_{\xi^k}$

$L_0 u = c\bar{u}$

So $\quad Lu = \bar{L}\bar{u} = \bar{L}_2\bar{u} + \bar{L}_1\bar{u} + \bar{L}_0\bar{u}$

the matrix $[\bar{a}^{kl}]$ for $\bar{L}_2$ is

$\cancel{J^T A J}$

CORRECTION JAJ^T

$J = \left[\dfrac{\partial \xi^r}{\partial x^s}\right]$ is the Jacobian matrix for $\xi(\vec{x})$.

entry$_{rs}$

entry$_{ij}(AB) = \sum_k a_{ik}b_{kj}$

entry$_{ij}(ABC) = \sum_{k,l} a_{ik}b_{kl}c_{lj}$

etc.

Step 3 Recall spectral theorem and quadratic forms from linear algebra

A is symmetric $\Rightarrow \exists\ \Theta$ orthogonal s.t.

$$\Theta^T A \Theta = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

diagonal matrix of eigenvalues of A

Θ is an orthonormal eigenbasis for A. $(\Theta^{-1} = \Theta^T)$

Thus $\cancel{\vec{\xi} = \Theta \vec{x}}$ (with $J = \Theta^T$)

correction
$\vec{\xi} = \Theta^T \vec{x}$
$\vec{x} = \Theta \vec{\xi}$

makes $\bar{L}_2 \bar{u} = \sum_{k=1}^{n} \lambda_k \bar{u}_{\xi^k \xi^k}$

now scale: $\xi^k = \begin{cases} z^k & \lambda_k = 0 \\ \sqrt{|\lambda_k|}\, z^k & \lambda_k \neq 0 \end{cases}$

$$\bar{u}(\vec{\xi}) = \bar{\bar{u}}(\vec{z})$$

for $\lambda_k \neq 0$, $\bar{\bar{u}}_{z^k} = \bar{u}_{\xi^k} \dfrac{\partial \xi^k}{\partial x^k}$ (no sum!)

$$= \sqrt{|\lambda_k|}\, \bar{u}_{\xi^k}$$

$$\bar{\bar{u}}_{z^k z^k} = |\lambda_k|\, \bar{u}_{\xi^k \xi^k}$$

So (after re-ordering variables)

$$\bar{\bar{L}}_2 \bar{\bar{u}} = \sum_{k=1}^{m} \bar{\bar{u}}_{z^k z^k} - \sum_{k=m+1}^{m+r} \bar{\bar{u}}_{z^k z^k}$$

canonical form for L

$m = \#$ pos evals of A $\quad$ $r = \#$ neg evals of A $\quad$ $n-m-r = \#$ zero evals of A

the list $(m, r, n-m-r)$ is called the <u>inertia</u> of quadratic form $Q(\vec{x}) = \vec{x}^T A \vec{x}$ and is invariant with respect to change of basis $\vec{x} = B\vec{\xi}$, which yields

$$\bar{Q}(\xi) = \xi^T (B^T A B) \xi,$$

m = dimension of maximum subspace on which Q is positive

r = dim of max. subs. on which Q is negative

$n-m-r$ = dim of max subspace on which Q is zero

Def $Lu = L_2 u + L_1 u + L_0 u$ is

a) elliptic at $\vec{x}_0$ if $\exists$ linear c.o.v. s.t. $\bar{L}_2 = \pm \left(\sum_{k=1}^{n} \bar{u}_{\xi^k \xi^k} \right)$ ~ all evals of A have same sign

b) hyperbolic at $\vec{x}_0$ if $\exists$ linear cov. s.t. $\bar{L}_2 = \pm \left(\bar{u}_{\xi^1 \xi^1} - \sum_{k=2}^{n} \bar{u}_{\xi^k \xi^k} \right)$ ~ $(n-1)$ evals of one sign, one of the opposite

c) parabolic at x_0 if $\exists$ linear cov s.t. $\bar{L}_2 = \pm \left(\sum_{k=1}^{n-1} \bar{u}_{\xi^k \xi^k} \right)$ ~ one zero eval. all others of same sign.

for $n \geq 3$ this does not exhaust all possibilities!

Homework for 9/24

2.8 #1. Also, if L is hyperbolic or parabolic, find all sol'ns to $Lu = 0$. If L is elliptic find a change of variables $\vec{\xi} = \vec{\xi}(\vec{x})$ s.t. $Lu = u_{\xi^1 \xi^1} + u_{\xi^2 \xi^2}$ (or use $\vec{\xi} = (\xi, \eta)$)

Do this extra work both ways we discussed:
the factoring method on Wednesday's notes, and
the rotation method (possibly followed by scaling) in today's notes.
Compare your results from these two methods.

3.11 #1, 2, 3

I may have some PDE derivation problems on Monday.

Example 1 revisited: On page 1 we found the general soltn of $u_{xx} + 2u_{xy} + u_{yy} = 0$.

It is $\boxed{u(x,y) = x\,q(y-x) + p(y-x)}$ where $p(\eta)$ and $q(\eta)$ are arbitrary C^2 functions.

Rework, using n-variable technique; with orthogonal COV and possibly scaling:

$$L = \partial_x^2 + 2\partial_x\partial_y + \partial_y^2 = \begin{bmatrix}\partial_x & \partial_y\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\partial_x\\ \partial_y\end{bmatrix}$$

$A = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$. $|A - \lambda I| = \lambda(\lambda - 2)$ so $\lambda = 0, 2$ are eigenvalues. Find evects by solving $\left[\begin{array}{cc|c}1-\lambda & 1 & 0\\ 1 & 1-\lambda & 0\end{array}\right]$

yields $A\begin{bmatrix}1\\1\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix}$, $A\begin{bmatrix}-1\\1\end{bmatrix} = 0\begin{bmatrix}1\\1\end{bmatrix}$

so $$A\overbrace{\begin{bmatrix}1/\sqrt2 & -1/\sqrt2\\ 1/\sqrt2 & 1/\sqrt2\end{bmatrix}}^{\Theta} = \left[\,2\begin{bmatrix}1/\sqrt2\\ 1/\sqrt2\end{bmatrix}\ 0\begin{bmatrix}-1/\sqrt2\\ 1/\sqrt2\end{bmatrix}\right] = \overbrace{\begin{bmatrix}1/\sqrt2 & -1/\sqrt2\\ 1/\sqrt2 & 1/\sqrt2\end{bmatrix}}^{\Theta}\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}; \quad A\Theta = \Theta\Lambda$$

$$\Theta^T A \Theta = \begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}$$

COV $J = \Theta^T$ $\quad \left[\dfrac{\partial \xi^i}{\partial x^j}\right] = \Theta^T \quad \begin{bmatrix}\xi\\ \eta\end{bmatrix} = \begin{bmatrix}1/\sqrt2 & 1/\sqrt2\\ -1/\sqrt2 & 1/\sqrt2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$

$\left[\dfrac{\partial x^i}{\partial \xi^j}\right] = \Theta \quad \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}1/\sqrt2 & -1/\sqrt2\\ 1/\sqrt2 & 1/\sqrt2\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix}.$

$\bar{L} = 2\bar{u}_{\xi\xi}$.

$\bar{L}\bar{u} = 0 \Rightarrow \bar{u}(\xi,\eta) = \tilde{p}(\eta) + \xi\,\tilde{q}(\eta)$

$$\Rightarrow \boxed{u(x,y) = \tilde{p}\left(\frac{y-x}{\sqrt2}\right) + \underbrace{\left(\frac{x+y}{\sqrt2}\right)}_{\left(\frac{y-x}{\sqrt2} + \sqrt2 x\right)}\tilde{q}\left(\frac{y-x}{\sqrt2}\right)}$$

$$\left(= \underbrace{\left[\tilde{p}\left(\frac{y-x}{\sqrt2}\right) + \left(\frac{y-x}{\sqrt2}\right)\tilde{q}\left(\frac{y-x}{\sqrt2}\right)\right]}_{:=\,p(y-x)} + \underbrace{\sqrt2\,x\,\tilde{q}\left(\frac{y-x}{\sqrt2}\right)}_{:=\,x\,q(y-x)}\right)$$

"same"!